1 Introduction

1.0 Introduction

Green's function is named after George Green for his fundamental contributions to potential theory, reciprocal relations, singular functions, and representative theorem. Green's functions can be extremely powerful in solving various differential equations and are also the essential components in the boundary integral equation method. As singular solutions to certain differential equations in their most generalized mathematical form, such solutions can find applications in nearly every field of science and engineering. They have been and will be continuously utilized in earth science/geophysics; civil, mechanical, and aerospace engineering; physics and material science; nanoscience/nanotechnology; biotechnology; information technology, and so forth. In this chapter, we define the Green's function and introduce its basic features, along with derivations of some of the common Green's functions in potential problems.

1.1 Definition of Green's Function

Green was a mathematician and physicist of United Kingdom (Cannell 2001; Challis and Sheard 2003). He not only developed this powerful tool for solving linear differential equations, but also contributed to various problems in elasticity. For instance, he offered a derivation of the governing equations of elasticity without using any hypothesis on the behavior of the molecular structure of the solids, and was able to show further that twenty-one elastic constants are required in general to account for the general anisotropy of elastic property (Timoshenko 1953). He further explained how symmetry can reduce the independent number of these constants.

Green's function is also called singular function, which is the fundamental solution of a (partial or ordinary) differential equation (or system of equations) in the problem domain (usually of infinite size) where the inhomogeneous term in the equation is replaced by the Dirac delta function.

As an example, let us consider the following differential equation in a two-dimensional (2D) or three-dimensional (3D) infinite and homogeneous domain,

$$Lu(\boldsymbol{r}) = f(\boldsymbol{r}) \tag{1.1}$$

where L is a general linear partial differential operator, f is the given inhomogeneous term, u is the function to be solved, and $\boldsymbol{r}$ is the plane or space position vector. The corresponding Green's function or the fundamental solution $G(\boldsymbol{r};\boldsymbol{r}_s)$ of Eq. (1.1) is then the solution of

$$LG(\boldsymbol{r};\boldsymbol{r}_s) = \delta(\boldsymbol{r} - \boldsymbol{r}_s) \tag{1.2}$$

where the Dirac delta function $\delta(\boldsymbol{r} - \boldsymbol{r}_s)$ means that when $\boldsymbol{r} = \boldsymbol{r}_s$ (i.e., the field point $\boldsymbol{r}$ coincides with the source point $\boldsymbol{r}_s$), it has an infinite value; otherwise, it equals zero. We should also mention that any domain integration of this delta function containing the source point $\boldsymbol{r}_s$ equals 1.

Once one derives the Green's function solution of Eq. (1.2), the solution of the corresponding inhomogeneous equation (1.1) can be simply expressed through the method of superposition. In other words, the solution of Eq. (1.1) can be expressed as

$$u(\boldsymbol{r}) = \int_V G(\boldsymbol{r}_s;\boldsymbol{r}) f(\boldsymbol{r}_s)\,\mathrm{d}V(\boldsymbol{r}_s) \tag{1.3}$$

Remark 1.1: In an infinite and homogeneous domain, the Green's function does not need to satisfy any boundary condition, except for perhaps certain constraints (i.e., the regular conditions) at infinity. For half-space, bimaterial space, and so forth, the Green's function is required to also satisfy the corresponding surface or interface conditions.

Remark 1.2: The Green's function depends on both the source and field points, thus being also called two-point function.

Remark 1.3: Attention should be paid to the relative location of the source and field points, except for the Green's function in a homogeneous infinite domain.

Remark 1.4: Relations (1.1)–(1.3) actually hold for any n-dimensional space.

Example 1: 2D (3D) Poisson's equations in an infinite domain.

The Poisson's equation in both 2D and 3D domains with an inhomogeneous term $f(\boldsymbol{r})$ can be expressed as

$$\nabla^2 u(\boldsymbol{r}) = f(\boldsymbol{r}) \tag{1.4}$$

where the Laplacian is defined as

$$\nabla^2 = \begin{cases} \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} & (\text{in 2D}\,(x,y)\text{-plane}) \\ \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} & (\text{in 3D}\,(x,y,z)\text{-space}) \end{cases} \tag{1.5}$$

The corresponding Green's function solutions to Eq. (1.4) (i.e., replacing the inhomogeneous term $f(\boldsymbol{r})$ by $\delta(\boldsymbol{r} - \boldsymbol{r}_s)$) are

$$G(\mathbf{r}-\mathbf{r}_s)=\begin{cases}\dfrac{-1}{2\pi}\ln\left(\dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}_s|}\right) & \text{(2D)}\\ \dfrac{-1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_s|} & \text{(3D)}\end{cases} \tag{1.6}$$

where $|\boldsymbol{r}-\boldsymbol{r}_s|$ is the distance between the field and source points (in both 2D and 3D domains). Then, based on Eq. (1.3), the particular solutions of the Poisson's equation (1.4) corresponding to the inhomogeneous "body force" term $f(\boldsymbol{r})$ can be found as

$$u(\boldsymbol{r})=\begin{cases}-\dfrac{1}{2\pi}\displaystyle\int_A f(\boldsymbol{r}_s)\ln\left(\dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}_s|}\right)\mathrm{d}A(\boldsymbol{r}_s) & \text{(2D)}\\ -\dfrac{1}{4\pi}\displaystyle\int_V \dfrac{f(\boldsymbol{r}_s)}{|\boldsymbol{r}-\boldsymbol{r}_s|}\mathrm{d}V(\boldsymbol{r}_s) & \text{(3D)}\end{cases} \tag{1.7}$$

Remark 1.5: The second integral expression in Eq. (1.7) for the 3D case was actually derived by Green, who introduced and solved the electric potential due to the density of the electricity (Green 1828).

Example 2: 2D (3D) Poisson formulae in a finite circle (sphere) with described potential on the boundary of $r = a$.

For this case, both the 2D and 3D problems can be described as

$$\begin{cases}\nabla^2 u = 0 & (r<a)\\ u|_{r=a} = \begin{cases} f(\theta) & \text{(2D)}\\ f(\theta,\varphi) & \text{(3D)}\end{cases}\end{cases} \tag{1.8}$$

where $f(\theta)$ ($f(\theta,\varphi)$ in 3D) is the potential given on the circle (sphere) in terms of θ (θ and φ in 3D). For 3D, θ is the zenithal angle measured from the positive z-axis, and φ is the azimuthal angle in the (x,y)-plane measured from the positive x-axis.

To find the potential within the circle (sphere in 3D) produced by the potential described on the boundary of the circle (sphere in 3D), one can make use of the Green's theorem in the following text (Eq. (1.20)). However, one will need the Green's function solution within the circle (sphere in 3D), which satisfies the zero-potential boundary condition (i.e., $G = 0$) on $r = a$. In other words, this special Green's function is the solution of the following boundary value problem:

$$\begin{cases}\nabla^2 G = \delta(\boldsymbol{r}-\boldsymbol{r}_s) & (r<a)\\ G|_{r=a} = 0\end{cases} \tag{1.9}$$

Fortunately, this Green's function solution within the circle (or sphere) under the zero-potential boundary condition can be found using the method of image. We take the 2D case as an example to show the process. As shown in Figure 1.1, the image point i of the inner source s is selected using the following relation

$$\overline{oi} = \frac{a^2}{\overline{os}} \tag{1.10a}$$

It can be shown that (Roach 1982) (Box 1.1)

$$\overline{oi} = \frac{a\,\overline{if}}{\overline{sf}} \tag{1.10b}$$

for the field point f located arbitrarily on the surface of the circle $r = a$. In terms of the position vectors, Eq. (1.10b) can be written as (after also moving all the quantities to the left-hand side and letting $r_i = |\boldsymbol{r}_i|$)

$$\frac{a|\boldsymbol{r} - \boldsymbol{r}_i|}{r_i|\boldsymbol{r} - \boldsymbol{r}_s|} = 1 \tag{1.11}$$

Box 1.1. Proof of Eq. (1.10b)

Let the angle between $\overline{of}$ and $\overline{oi}$ or $\overline{os}$ as α, then we have

$$\begin{aligned} \overline{if}^2 &= \overline{of}^2 + \overline{oi}^2 - 2\overline{of}\,\overline{oi}\cos\alpha \\ \overline{sf}^2 &= \overline{of}^2 + \overline{os}^2 - 2\overline{of}\,\overline{os}\cos\alpha \end{aligned}$$

Making use of Eq. (1.10a) and noticing that $\overline{of} = a$, then Eq. (1.10b) holds.

Because the full-plane potential Green's function is in the form of ln(r) as can be observed from Eq. (1.6), the Green's function based on the left-hand side of Eq. (1.11) (taking the natural logarithm of it) will satisfy the zero-potential boundary condition on the circle of $r = a$. In other words, the Green's function solution of Eq. (1.9) can be expressed as

$$G(\boldsymbol{r};\boldsymbol{r}_s) = \frac{-1}{2\pi}\ln\left(\frac{a|\boldsymbol{r} - \boldsymbol{r}_i|}{r_i|\boldsymbol{r} - \boldsymbol{r}_s|}\right) \tag{1.12}$$

which can be written alternatively as

$$G(\boldsymbol{r};\boldsymbol{r}_s) = \frac{-1}{2\pi}\ln\left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_s|}\right) + \frac{1}{2\pi}\ln\left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_i|}\right) + \frac{-1}{2\pi}\ln\left(\frac{a}{r_i}\right) \tag{1.13}$$

This is the Green's function solution of Eq. (1.9), which holds for both the source and field points at any location within the circle $r = a$ (except for $\boldsymbol{r} = \boldsymbol{r}_s$). It is observed from Eq. (1.13) that the first term on the right-hand side is the original source Green's function, the second term the image source Green's function, and the third term a constant added to satisfy the zero-potential boundary condition on the circle $r = a$.

Then, applying this special Green's function within the circle to the Green's theorem Eq. (1.20), and making use of the boundary condition of the potential function u on the circle $r = a$, we have (let $\phi = u$, $\psi = G$ in Eq. (1.20))

$$u(r_s, \theta_s) = \int_C f(\theta) q_r(\boldsymbol{r}; r_s, \theta_s)\,\mathrm{d}C(\boldsymbol{r}) \tag{1.14}$$

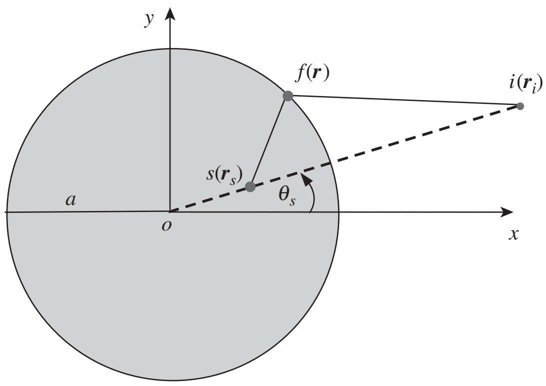

Figure 1.1. A circle of radius a, which is under a point source at s (with coordinate $\boldsymbol{r}_s$) within the circle. The field point at f (with coordinate $\boldsymbol{r}$) for the special case shown in the figure is on the circle, and the image point of the inner source s with respect to the circle is at i (with coordinate $\boldsymbol{r}_i$).

where the source point $\boldsymbol{r}_s$ is replaced by its polar coordinates (r_s,θ_s), the integral on the right-hand side is on the circle $r = a$, and q_r is the normal derivative of the Green's function at $\boldsymbol{r}$, which in our case, equals $\partial G/\partial r$. Expressing the field point $\boldsymbol{r}$ in terms of the polar coordinate (r,θ) in the Green's function (1.13), taking its derivative with respect to r, and substituting the result into Eq. (1.14), one can finally arrive at the following famous 2D Poisson integral formula

$$u(r_s,\theta_s) = \frac{a^2 - r_s^2}{2\pi}\int_0^{2\pi} \frac{f(\theta)}{a^2 + r_s^2 - 2ar_s\cos(\theta - \theta_s)}\mathrm{d}\theta \tag{1.15}$$

This represents the solution of the potential at any inner point (r_s,θ_s) due to a prescribed potential function $f(\theta)$ on the circle $r = a$, as described by Eq. (1.8).

For the corresponding 3D case, Eq. (1.11) can be equivalently expressed as, for the field point f at any location on the spherical surface $r = a$ (thinking of Figure 1.1 being a 3D diagram)

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_s|} - \frac{r_i}{a}\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_i|} = 0 \tag{1.16}$$

Thus the 3D Green's function of Eq. (1.9) within the sphere $r = a$ (with both the source and field points within the sphere, except for $\boldsymbol{r} = \boldsymbol{r}_s$) can be constructed as

$$G(\boldsymbol{r};\boldsymbol{r}_s) = \frac{-1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_s|} + \frac{r_i}{4\pi a}\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_i|} \tag{1.17}$$

Similarly, applying the Green's theorem with one of the functions being this Green's function and the other being the real problem as described by Eq. (1.8), we find the following classic 3D Poisson integral formula (using the spherical coordinates (r_s,θ_s,φ_s) for the source point $\boldsymbol{r}_s$)

$$u(r_s,\theta_s,\varphi_s) = \frac{a(a^2 - r_s^2)}{4\pi}\int_0^{\pi}\sin\theta\mathrm{d}\theta\int_0^{2\pi}\frac{f(\theta,\varphi)}{(a^2 + r_s^2 - 2ar_s\cos\gamma)^{3/2}}\mathrm{d}\varphi \tag{1.18}$$

where

$$\cos\gamma = \cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos(\varphi - \varphi_s) \tag{1.19}$$

Remark 1.6: Notice the following relations when deriving Eq. (1.18)

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \\ \mathrm{d}S &= r^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \end{aligned}$$

1.2 Green's Theorems and Identities

One of the Green's theorems is also called the divergence theorem, Gauss's theorem, or Green's second identity. For any two functions that are twice differentiable with respect to the 3D coordinates (x,y,z), the following identity holds:

$$\int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot \mathrm{d}\boldsymbol{S} = \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\mathrm{d}V \tag{1.20}$$

where S is the boundary of the defined volume V, $\mathrm{d}\boldsymbol{S}$ is an oriented area element on S, and $\nabla = \boldsymbol{i}_x\partial_x + \boldsymbol{i}_y\partial_y + \boldsymbol{i}_z\partial_z$ is the gradient operator vector in the Cartesian coordinate system. Equation (1.20) builds the important relation between the volume integral of the functions and the boundary (surface) integral of these functions. As such, it has the following two important applications:

(1) It can be applied to carry out the volume integral on the right hand side if the surface integral on the left-hand side is computationally more efficient and convenient than the volume integral. In other words, one does not need to carry out the complicated volume integral. Similarly, if the involved surface integral on the left-hand side is difficult to be carried out, one can just calculate the volume integral on the right-hand side.
(2) One can express one of the terms in terms of the other three if the latter three can be evaluated efficiently. This is particularly important in the boundary integral equation method.

If we replace one of them in this equation by the 3D Green's function $-1/(4\pi r)$, then the solution for the other potential can be expressed in terms of its value (potential and its derivative) on the surface, as will be discussed in Section 1.4.

Remark 1.7: The Green's theorem (1.20) can be derived through the simple reciprocal relations or reciprocity that Green derived for electrostatic problems.

Green (1828) also derived the Green's first identity, which is expressed as

$$\int_S \phi\boldsymbol{\nabla}\psi\cdot\mathrm{d}\boldsymbol{S} = \int_V (\phi\nabla^2\psi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi)\mathrm{d}V \tag{1.21}$$

It is noted that Eq. (1.20) can be simply derived from Eq. (1.21).

The special case of Eq. (1.20) is when one of the functions, say ψ is constant. Then Eq. (1.20) is reduced to

$$\int_S \nabla\phi \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla^2\phi \mathrm{d}V \tag{1.22}$$

Or in terms of their components in 3D

$$\int_S \left[\frac{\partial\phi}{\partial x}\mathrm{d}y\mathrm{d}z + \frac{\partial\phi}{\partial y}\mathrm{d}z\mathrm{d}x + \frac{\partial\phi}{\partial z}\mathrm{d}x\,\mathrm{d}y\right] = \int_V \left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right]\mathrm{d}V \tag{1.23}$$

Equations (1.20) to (1.23) are the Green's relations between the volume integral V and the surface integral S which bounds the volume V. There are also some corresponding Green's relations in 2D that relate a 2D area A to its closed boundary C, which are discussed in the following text.

The first 2D relation (say in the (x,y)-plane) is similar to the 3D divergence theorem. We let C be a positively oriented, piecewise smooth, simple, closed curve, with its outward normal being $\boldsymbol{n}$. We further let A be a region within the flat (x,y)-plane enclosed by the curve C. If the vector function $\boldsymbol{u}$ has continuous first-order partial derivatives in A, then the following relation holds

$$\oint_C \boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}C = \int_A \nabla\cdot\boldsymbol{u}\mathrm{d}A \tag{1.24}$$

Or in terms of their components

$$\oint_C \left(u_x\mathrm{d}y - u_y\mathrm{d}x\right) = \int_A \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)\mathrm{d}A \tag{1.25}$$

Equation (1.24) or (1.25) relates the closed line integral to the integral over the area enclosed by the line. It relates the normal component of a 2D vector function on one side to its divergence on the other side of the equation.

The second relation between a 3D curve and a 3D surface is also called the Green's theorem or the Stokes's theorem, where the tangential component of the 3D vector function on one side is related to its vector curl on the other side of the equation. In other words,

$$\oint_C \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{C} = \int_S (\nabla\times\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{S} \tag{1.26}$$

where the oriented curve C is defined to be positive when one travels along C in the positive direction while the enclosed surface S is on the left-hand side of the traveler. It should be pointed out that the curve C is generally 3D (i.e., it is not required to be in a given flat plane), and that even the curve C is in a flat plane, the surface S enclosed can still be in 3D.

It is noted that in terms of their components, Eq. (1.26) can be expressed as

$$\oint_C u_k\mathrm{d}x_k = \varepsilon_{kij}\int_S \partial_i u_j \mathrm{d}S_k \tag{1.27}$$

where ε_{ijk} is the permutation tensor (=1 for $(ijk) \in \{(123), (231), (312)\}$; = −1 for $(ijk) \in \{(321), (132), (213)\}$; = 0 if $i = j$, or $j = k$, or $k = i$). Summation convention is implied over the repeated index.

If the curve C and its enclosed surface A are both in a flat plane, say the (x,y)-plane, then the Stokes' theorem (Eq. (1.26) or (1.27)) is reduced to

$$\oint_C \left(u_x \mathrm{d}x + u_y \mathrm{d}y\right) = \int_A \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \mathrm{d}A \tag{1.28}$$

This compares with the 2D divergence relation Eq. (1.25).

Remark 1.8: In solving various engineering and physics problems, one often ends up with an integral expression as the problem solution. The Green's theorems and identities (Eqs. (1.20)–(1.28)) provide us with the opportunity of converting the integral into forms that could be integrated easily.

Remark 1.9: If no analytical integration exists for the involved integral, one at least can convert the integral into one with lower dimensions. For instance, one can reduce a 3D integral to 2D, or a 2D integral to 1D. The integral with lower dimension is computationally more efficient. Some of these relations have been frequently applied in various boundary integral equation formulations.

1.3 Green's Functions of Potential Problems

1.3.1 Primary on 2D and 3D Potential Green's Functions

The 2D or 3D potential function u is mathematically only a scalar function with certain features; however, it can be related to so many engineering and physics problems, besides the well-known electric and magnetic potentials. This function can be a quantity associated with elastic torsion, antiplane deformation, and so forth. It can be also the seepage, aquifer, heat conduction, diffusion, and fluid motion. As such, a basic understanding of the Green's functions of potential problems is important.

The full-plane (or full-space) Green's function of the potential-type problem satisfies the following governing equation

$$k\nabla^2 G = \delta(\boldsymbol{r}) \tag{1.29}$$

in the whole infinite domain (either 2D or 3D). The physical meaning of the coefficient k is attached to the physics of the potential field u. For instance, it will be the thermal conductivity if u represents the temperature change.

The Green's function G is defined as such that for $\boldsymbol{r} \neq \boldsymbol{0}$, it is the solution of the following equation

$$k\nabla^2 G = 0 \tag{1.30}$$

and that it also satisfies the following integral relation around the source point $\boldsymbol{r} = \boldsymbol{0}$, due to the property of the delta function δ,

$$\int_{V_\varepsilon} k\nabla^2 G \mathrm{d}V = 1 \tag{1.31}$$

where V_ε is a circle (sphere) of arbitrary radius ε centered at the source point $\boldsymbol{r} = \boldsymbol{0}$. The way to find the Green's function is to first solve the homogeneous Eq. (1.30) and then apply the condition (1.31) to determine the unknown coefficients, keeping in mind that the radius ε is of arbitrary length. By following these steps (Brebbia and Dominguez, 1996), the Green's function of Eq. (1.29) at the field point $\boldsymbol{r}$ when the source is applied at the origin is found to be

$$G(\boldsymbol{r}) = \begin{cases} \dfrac{-1}{2\pi k}\ln\dfrac{1}{r} & \text{(2D)} \\ \dfrac{-1}{4\pi k r} & \text{(3D)} \end{cases} \tag{1.32}$$

where $r = |\boldsymbol{r} - \boldsymbol{0}| = |\boldsymbol{r}|$ is the distance of the field point to the source point at the origin. This Green's function solution is essentially the same as that in Eq. (1.6) when the material property parameter is normalized to $k = 1$.

Remark 1.10: Adding an arbitrary 2D/3D harmonic function to the right-hand side of Eq. (1.32) still gives us the Green's function solution of Eq. (1.29). A special case is that the harmonic function is a constant, say, equal to b. Then an alternative Green's function solution of Eq. (1.29) is

$$G(\boldsymbol{r}) = \begin{cases} \dfrac{-1}{2\pi k}\ln\dfrac{1}{r} + b & \text{(2D)} \\ \dfrac{-1}{4\pi k r} + b & \text{(3D)} \end{cases} \tag{1.33}$$

In other words, the potential Green's function could be unique up to an arbitrary harmonic function.

1.3.2 Potential Green's Functions in Bimaterial Planes

The method of image presented in this chapter for the potential problem in a circle (sphere) domain can be utilized to find the potential Green's functions in bimaterial planes/spaces. We start with the bimaterial plane case first. We assume that in the $y > 0$ half-plane (with material property $k^{(1)}$), there is a source of unit magnitude applied at $(x_s, y_s > 0)$, and the $y < 0$ half-plane (with material property $k^{(2)}$) is free of any source. Along the interface $y = 0$, we assume the perfect interface condition, that is, both the potential G and its normal flux $q_y = k\partial G/\partial y$ are continuous. As such, the governing equations are

$$\begin{aligned} &k^{(1)}\nabla^2 G^{(1)} = \delta(x - x_s)\delta(y - y_s) \qquad (y, y_s > 0) \\ &k^{(2)}\nabla^2 G^{(2)} = 0 \qquad\qquad\qquad\qquad\; (y < 0) \end{aligned} \tag{1.34}$$

For the perfect interface continuity conditions on $y = 0$, we have

$$G^{(1)} = G^{(2)}, \qquad q_y^{(1)} = q_y^{(2)} \tag{1.35}$$

The Green's function solutions at any field point (x,y) under this perfect interface condition are

$$G(x,y;x_s,y_s)=\begin{cases}\dfrac{-1}{2\pi k^{(1)}}\ln\dfrac{1}{r_1}+\dfrac{-(k^{(1)}-k^{(2)})}{2\pi k^{(1)}(k^{(1)}+k^{(2)})}\ln\dfrac{1}{r_2} & (y>0)\\ \dfrac{-1}{\pi(k^{(1)}+k^{(2)})}\ln\dfrac{1}{r_1} & (y<0)\end{cases}\tag{1.36}$$

where

$$\begin{aligned}r_1&=\sqrt{(x-x_s)^2+(y-y_s)^2}\\ r_2&=\sqrt{(x-x_s)^2+(y+y_s)^2}\end{aligned}\tag{1.37}$$

are, respectively, the distance between the field point (x,y) and the source (x_s,y_s) and the distance between the field point and the image source $(x_s,-y_s)$.

Remark 1.11: The results for the source in the lower half-plane can be found by simply switching "(1)" and "(2)" attached to the material coefficient k, and switching the subscripts between 1 and 2 of the distance r.

Remark 1.12: When the material properties of the two half-planes are identical (i.e., $k^{(1)} = k^{(2)} = k$), Eq. (1.36) is reduced to the Green's function in the full-plane with material property k.

Remark 1.13: When $k^{(2)} = 0$, we have the flux-free boundary condition at $y = 0$ for the upper half-plane, and the corresponding half-plane solution with material property $k^{(1)} = k$ in the $y > 0$ half-plane is

$$G(x,y;x_s,y_s)=\frac{-1}{2\pi k}\ln\frac{1}{r_1 r_2}\tag{1.38}$$

Remark 1.14: When $G = 0$ on the boundary $y = 0$ (i.e., $k^{(2)}$ approaches infinity), the corresponding half-plane solution in $y > 0$ is

$$G(x,y;x_s,y_s)=\frac{-1}{2\pi k}\ln\frac{r_2}{r_1}\tag{1.39}$$

Clearly, the preceding Green's function (1.39) can be used to construct the following Poisson integral formula

$$u(x_s,y_s)=\frac{y_s}{\pi k}\int_{-\infty}^{\infty}\frac{f(x)}{(x-x_s)^2+y_s^2}\,\mathrm{d}x\tag{1.40}$$

with $f(x) = u(x,0)$ being prescribed on $y = 0$. Thus, the potential function u, which satisfies the 2D Laplace equation and is known on $y = 0$, can be calculated from Eq. (1.40) so that its value at any interior point in the upper half-plane can be obtained.

Remark 1.15: It should note here that, the integral in Eq. (1.40) should be over a closed contour, say with an additional large semicircle of radius R. It can be shown that, to eliminate the integral over the semicircle, the potential prescribed on the boundary should satisfy the condition $u = O(R^{\alpha})$ with $\alpha < 1$ as $R\to\infty$ (Greenberg 1971).