

## PRINCIPLES OF DISCRETE TIME MECHANICS

Could time be discrete on some unimaginably small scale? Exploring the idea in depth, this unique introduction to discrete time mechanics systematically builds the theory up from scratch, beginning with the historical, physical and mathematical background to the chronon hypothesis.

Covering classical and quantum discrete time mechanics, this book presents all the tools needed to formulate and develop applications of discrete time mechanics in a number of areas, including spreadsheet mechanics, classical and quantum register mechanics, and classical and quantum mechanics and field theories. A consistent emphasis on contextuality and the observer–system relationship is maintained throughout.

GEORGE JAROSZKIEWICZ is an Associate Professor at the School of Mathematical Sciences, University of Nottingham, having formerly held positions at the University of Oxford and the University of Kent.

## CAMBRIDGE MONOGRAPHS ON MATHEMATICAL PHYSICS

General Editors: P. V. Landshoff, D. R. Nelson, S. Weinberg

- S. J. Aarseth *Gravitational N-Body Simulations: Tools and Algorithms*  
 J. Ambjørn, B. Durhuus and T. Jonsson *Quantum Geometry: A Statistical Field Theory Approach*  
 A. M. Anile *Relativistic Fluids and Magneto-fluids: With Applications in Astrophysics and Plasma Physics*  
 J. A. de Azcárraga and J. M. Izquierdo *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics*<sup>†</sup>  
 O. Babelon, D. Bernard and M. Talon *Introduction to Classical Integrable Systems*  
 F. Bastianelli and P. van Nieuwenhuizen *Path Integrals and Anomalies in Curved Space*  
 V. Belinski and E. Verdaguer *Gravitational Solitons*  
 J. Bernstein *Kinetic Theory in the Expanding Universe*  
 G. F. Bertsch and R. A. Broglia *Oscillations in Finite Quantum Systems*  
 N. D. Birrell and P. C. W. Davies *Quantum Fields in Curved Space*<sup>†</sup>  
 K. Bolejko, A. Krasinski, C. Hellaby and M.-N. Célérier *Structures in the Universe by Exact Methods: Formation, Evolution, Interactions*  
 D. M. Brink *Semi-Classical Methods for Nucleus-Nucleus Scattering*<sup>†</sup>  
 M. Burgess *Classical Covariant Fields*  
 E. A. Calzetta and B.-L. B. Hu *Nonequilibrium Quantum Field Theory*  
 S. Carlip *Quantum Gravity in 2+1 Dimensions*<sup>†</sup>  
 P. Cartier and C. DeWitt-Morette *Functional Integration: Action and Symmetries*  
 J. C. Collins *Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion*<sup>†</sup>  
 P. D. B. Collins *An Introduction to Regge Theory and High Energy Physics*<sup>†</sup>  
 M. Creutz *Quarks, Gluons and Lattices*<sup>†</sup>  
 P. D. D'Eath *Supersymmetric Quantum Cosmology*  
 J. Dereziński and C. Gérard *Mathematics of Quantization and Quantum Fields*  
 F. de Felice and D. Bini *Classical Measurements in Curved Space-Times*  
 F. de Felice and C. J. S. Clarke *Relativity on Curved Manifolds*  
 B. DeWitt *Supermanifolds, 2nd edition*  
 P. G. O. Freund *Introduction to Supersymmetry*<sup>†</sup>  
 F. G. Friedlander *The Wave Equation on a Curved Space-Time*<sup>†</sup>  
 J. L. Friedman and N. Stergioulas *Rotating Relativistic Stars*  
 Y. Frishman and J. Sonnenschein *Non-Perturbative Field Theory: From Two Dimensional Conformal Field Theory to QCD in Four Dimensions*  
 J. A. Fuchs *Affine Lie Algebras and Quantum Groups: An Introduction, with Applications in Conformal Field Theory*<sup>†</sup>  
 J. Fuchs and C. Schweigert *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists*<sup>†</sup>  
 Y. Fujii and K. Maeda *The Scalar-Tensor Theory of Gravitation*  
 J. A. H. Futterman, F. A. Handler, R. A. Matzner *Scattering from Black Holes*<sup>†</sup>  
 A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev *Harmonic Superspace*  
 R. Gambini and J. Pullin *Loops, Knots, Gauge Theories and Quantum Gravity*<sup>†</sup>  
 T. Gannon *Moonshine beyond the Monster: The Bridge Connecting Algebra, Modular Forms and Physics*  
 M. Göckeler and T. Schücker *Differential Geometry, Gauge Theories, and Gravity*<sup>†</sup>  
 C. Gómez, M. Ruiz-Altaba and G. Sierra *Quantum Groups in Two-Dimensional Physics*  
 M. B. Green, J. H. Schwarz and E. Witten *Superstring Theory Volume 1: Introduction*  
 M. B. Green, J. H. Schwarz and E. Witten *Superstring Theory Volume 2: Loop Amplitudes, Anomalies and Phenomenology*  
 V. N. Gribov *The Theory of Complex Angular Momenta: Gribov Lectures on Theoretical Physics*  
 J. B. Griffiths and J. Podolský *Exact Space-Times in Einstein's General Relativity*<sup>†</sup>  
 S. W. Hawking and G. F. R. Ellis *The Large Scale Structure of Space-Time*<sup>†</sup>  
 F. Iachello and A. Arima *The Interacting Boson Model*  
 F. Iachello and P. van Isacker *The Interacting Boson-Fermion Model*  
 C. Itzykson and J.-M. Drouffe *Statistical Field Theory Volume 1: From Brownian Motion to Renormalization and Lattice Gauge Theory*<sup>†</sup>  
 C. Itzykson and J. M. Drouffe *Statistical Field Theory Volume 2: Strong Coupling, Monte Carlo Methods, Conformal Field Theory and Random Systems*<sup>†</sup>  
 G. Jaroszkiewicz, *Principles of Discrete Time Mechanics*  
 C. V. Johnson *D-Branes*  
 P. S. Joshi *Gravitational Collapse and Spacetime Singularities*<sup>†</sup>  
 J. I. Kapusta and C. Gale *Finite-Temperature Field Theory: Principles and Applications, 2nd edition*

- V. E. Korepin, N. M. Bogoliubov and A. G. Izergin *Quantum Inverse Scattering Method and Correlation Functions*<sup>†</sup>  
 M. Le Bellac *Thermal Field Theory*<sup>†</sup>  
 Y. Makeenko *Methods of Contemporary Gauge Theory*  
 N. Manton and P. Sutcliffe *Topological Solitons*  
 N. H. March *Liquid Metals: Concepts and Theory*  
 I. Montvay and G. Münster *Quantum Fields on a Lattice*<sup>†</sup>  
 L. O’Raifeartaigh *Group Structure of Gauge Theories*<sup>†</sup>  
 T. Ortín *Gravity and Strings*  
 A. M. Ozorio de Almeida *Hamiltonian Systems: Chaos and Quantization*<sup>†</sup>  
 L. Parker and D. Toms *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*  
 R. Penrose and W. Rindler *Spinors and Space-Time Volume 1: Two-Spinor Calculus and Relativistic Fields*<sup>†</sup>  
 R. Penrose and W. Rindler *Spinors and Space-Time Volume 2: Spinor and Twistor Methods in Space-Time Geometry*<sup>†</sup>  
 S. Pokorski *Gauge Field Theories, 2nd edition*<sup>†</sup>  
 J. Polchinski *String Theory Volume 1: An Introduction to the Bosonic String*  
 J. Polchinski *String Theory Volume 2: Superstring Theory and Beyond*  
 J. C. Polkinghorne *Models of High Energy Processes*<sup>†</sup>  
 V. N. Popov *Functional Integrals and Collective Excitations*<sup>†</sup>  
 L. V. Prokhorov and S. V. Shabanov *Hamiltonian Mechanics of Gauge Systems*  
 A. Recknagel and V. Schomerus, *Boundary Conformal Field Theory and the Worldsheet Approach to D-Branes*  
 R. J. Rivers *Path Integral Methods in Quantum Field Theory*<sup>†</sup>  
 R. G. Roberts *The Structure of the Proton: Deep Inelastic Scattering*<sup>†</sup>  
 C. Rovelli *Quantum Gravity*<sup>†</sup>  
 W. C. Saslaw *Gravitational Physics of Stellar and Galactic Systems*<sup>†</sup>  
 R. N. Sen *Causality, Measurement Theory and the Differentiable Structure of Space-Time*  
 M. Shifman and A. Yung *Supersymmetric Solitons*  
 H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt *Exact Solutions of Einstein’s Field Equations, 2nd edition*<sup>†</sup>  
 J. Stewart *Advanced General Relativity*<sup>†</sup>  
 J. C. Taylor *Gauge Theories of Weak Interactions*<sup>†</sup>  
 T. Thiemann *Modern Canonical Quantum General Relativity*  
 D. J. Toms *The Schwinger Action Principle and Effective Action*<sup>†</sup>  
 A. Vilenkin and E. P. S. Shellard *Cosmic Strings and Other Topological Defects*  
 R. S. Ward and R. O. Wells, Jr *Twistor Geometry and Field Theory*  
 E. J. Weinberg *Classical Solutions in Quantum Field Theory: Solitons and Instantons in High Energy Physics*  
 J. R. Wilson and G. J. Mathews *Relativistic Numerical Hydrodynamics*

<sup>†</sup> Issued as a paperback

# Principles of Discrete Time Mechanics

GEORGE JAROSZKIEWICZ

*University of Nottingham*



**CAMBRIDGE**  
UNIVERSITY PRESS

Cambridge University Press & Assessment  
978-1-107-03429-7 — Principles of Discrete Time Mechanics  
George Jaroszkiewicz  
Frontmatter  
[More Information](#)

---



**CAMBRIDGE**  
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,  
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of  
education, learning and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9781107034297](http://www.cambridge.org/9781107034297)

© G. Jaroszkiewicz 2014

This publication is in copyright. Subject to statutory exception and to the provisions  
of relevant collective licensing agreements, no reproduction of any part may take  
place without the written permission of Cambridge University Press & Assessment.

First published 2014

*A catalogue record for this publication is available from the British Library*

ISBN 978-1-107-03429-7 Hardback

Cambridge University Press & Assessment has no responsibility for the persistence  
or accuracy of URLs for external or third-party internet websites referred to in this  
publication and does not guarantee that any content on such websites is, or will  
remain, accurate or appropriate.

Contents

<i>Preface</i>	<i>page</i> xiii
 <b>Part I Discrete time concepts</b>	
<b>1 Introduction</b>	<b>3</b>
1.1 What is time?	3
1.2 The architecture of time	5
1.3 The chronon: historical perspectives	14
1.4 The chronon: some modern perspectives	17
1.5 Plan of this book	23
 <b>2 The physics of discreteness</b>	 <b>24</b>
2.1 The natural occurrence of discreteness	24
2.2 Fourier-transform scales	25
2.3 Atomic scales of time	27
2.4 De Broglie scales	28
2.5 Hadronic scales	30
2.6 Grand unified scales	30
2.7 Planck scales	31
 <b>3 The road to calculus</b>	 <b>32</b>
3.1 The origins of calculus	32
3.2 The infinitesimal calculus and its variants	37
3.3 Non-standard analysis	40
3.4 q-Calculus	41
 <b>4 Temporal discretization</b>	 <b>46</b>
4.1 Why discretize time?	46
4.2 Notation	47
4.3 Some useful results	50
4.4 Discrete analogues of some generalized functions	52
4.5 Discrete first derivatives	53
4.6 Difference equations	55
4.7 Discrete Wronskians	58

<b>5</b>	<b>Discrete time dynamics architecture</b>	<b>61</b>
5.1	Mappings, functions	61
5.2	Generalized sequences	65
5.3	Causality	66
5.4	Discrete time	67
5.5	Second-order architectures	69
<b>6</b>	<b>Some models</b>	<b>71</b>
6.1	Reverse engineering solutions	71
6.2	Reverse engineering constants of the motion	74
6.3	First-order discrete time causality	75
6.4	The Laplace-transform method	78
<b>7</b>	<b>Classical cellular automata</b>	<b>80</b>
7.1	Classical cellular automata	80
7.2	One-dimensional cellular automata	82
7.3	Spreadsheet mechanics	85
7.4	The Game of Life	88
7.5	Cellular time dilation	89
7.6	Classical register mechanics	97
 <b>Part II Classical discrete time mechanics</b>		
<b>8</b>	<b>The action sum</b>	<b>111</b>
8.1	Configuration-space manifolds	111
8.2	Continuous time action principles	112
8.3	The discrete time action principle	117
8.4	The discrete time equations of motion	119
8.5	The discrete time Noether theorem	119
8.6	Conserved quantities via the discrete time Weiss action principle	121
<b>9</b>	<b>Worked examples</b>	<b>122</b>
9.1	The complex harmonic oscillator	122
9.2	The anharmonic oscillator	124
9.3	Relativistic-particle models	126
<b>10</b>	<b>Lee's approach to discrete time mechanics</b>	<b>129</b>
10.1	Lee's discretization	129
10.2	The standard particle system	131
10.3	Discussion	133
10.4	Return to the relativistic point particle	134

*Contents*

ix

<b>11 Elliptic billiards</b>	<b>136</b>
11.1 The general scenario	136
11.2 Elliptic billiards via the geometrical approach	137
11.3 Elliptic billiards via Lee mechanics	140
11.4 Complex-plane billiards	142
<b>12 The construction of system functions</b>	<b>144</b>
12.1 Phase space	144
12.2 Hamilton's principal function	145
12.3 Virtual-path construction of system functions	148
<b>13 The classical discrete time oscillator</b>	<b>151</b>
13.1 The discrete time oscillator	151
13.2 The Newtonian oscillator	152
13.3 Temporal discretization of the Newtonian oscillator	153
13.4 The generalized oscillator	154
13.5 Solutions	154
13.6 The three regimes	155
13.7 The Logan invariant	156
13.8 The oscillator in three dimensions	157
13.9 The anharmonic oscillator	158
<b>14 Type-2 temporal discretization</b>	<b>160</b>
14.1 Introduction	160
14.2 q-Mechanics	161
14.3 Phi-functions	164
14.4 The phi-derivative	165
14.5 Phi-integrals	166
14.6 The summation formula	166
14.7 Conserved currents	168
<b>15 Intermission</b>	<b>170</b>
15.1 The continuous time Lagrangian approach	171
15.2 The discrete time Lagrangian approach	173
15.3 Extended discrete time mechanics	175
<b>Part III Discrete time quantum mechanics</b>	
<b>16 Discrete time quantum mechanics</b>	<b>181</b>
16.1 Quantization	181
16.2 Quantum dynamics	184
16.3 The Schrödinger picture	185



x	<i>Contents</i>	
16.4	Position eigenstates	185
16.5	Normal-coordinate systems	188
16.6	Compatible operators	190
<b>17</b>	<b>The quantized discrete time oscillator</b>	<b>192</b>
17.1	Introduction	192
17.2	Canonical quantization	193
17.3	The inhomogeneous oscillator	197
17.4	The elliptic regime	199
17.5	The hyperbolic regime	202
17.6	The time-dependent oscillator	203
<b>18</b>	<b>Path integrals</b>	<b>209</b>
18.1	Introduction	209
18.2	Feynman’s path integrals	209
18.3	Lee’s path integral	215
<b>19</b>	<b>Quantum encoding</b>	<b>217</b>
19.1	Introduction	217
19.2	First-order quantum encoding	218
19.3	Second-order quantum encoding	220
19.4	Invariants of the motion	221
<b>Part IV Discrete time classical field theory</b>		
<b>20</b>	<b>Discrete time classical field equations</b>	<b>227</b>
20.1	Introduction	227
20.2	System functions for discrete time field theories	227
20.3	System functions for node variables	228
20.4	Equations of motion for node variables	230
20.5	Exact and near symmetry invariants	231
20.6	Linear momentum	233
20.7	Orbital angular momentum	234
20.8	Link variables	234
<b>21</b>	<b>The discrete time Schrödinger equation</b>	<b>236</b>
21.1	Introduction	236
21.2	Stationary states	239
21.3	Vibrancy relations	242
21.4	Linear independence and inner products	242
21.5	Conservation of charge	244

*Contents*

xi

<b>22 The discrete time Klein–Gordon equation</b>	<b>246</b>
22.1 Introduction	246
22.2 Linear momentum	248
22.3 Orbital angular momentum	249
22.4 The free-charged Klein–Gordon equation	250
<b>23 The discrete time Dirac equation</b>	<b>253</b>
23.1 Introduction	253
23.2 Grassmann variables in mechanics	254
23.3 The Grassmannian oscillator in continuous time	256
23.4 The Grassmannian oscillator in discrete time	258
23.5 The discrete time free Dirac equation	260
23.6 Charge and charge density	262
<b>24 Discrete time Maxwell equations</b>	<b>265</b>
24.1 Classical electrodynamical fields	265
24.2 Gauge invariance	267
24.3 The inhomogeneous equations	269
24.4 The charge-free equations	270
24.5 Gauge transformations and virtual paths	271
24.6 Coupling to matter fields	272
<b>25 The discrete time Skyrme model</b>	<b>275</b>
25.1 The Skyrme model	275
25.2 The $SU(2)$ particle	277
25.3 The $\sigma$ model	281
25.4 Further considerations	282
<b>Part V Discrete time quantum field theory</b>	
<b>26 Discrete time quantum field theory</b>	<b>287</b>
26.1 Introduction	287
26.2 The discrete time free quantized scalar field	289
26.3 The discrete time free quantized Dirac field	292
26.4 The discrete time free quantized Maxwell fields	297
<b>27 Interacting discrete time scalar fields</b>	<b>306</b>
27.1 Reduction formulae	307
27.2 Interacting fields: scalar field theory	308
27.3 Feynman rules for discrete time-ordered products	310
27.4 The two–two box scattering diagram	313
27.5 The vertex functions	316

xii	<i>Contents</i>	
27.6	The propagators	316
27.7	Rules for scattering amplitudes	318
 <b>Part VI Further developments</b>		
<b>28</b>	<b>Space, time and gravitation</b>	<b>323</b>
28.1	Snyder’s quantized spacetime	323
28.2	Discrete time quantum fields on Robertson–Walker spacetimes	328
28.3	Regge calculus	331
<b>29</b>	<b>Causality and observation</b>	<b>333</b>
29.1	Introduction	333
29.2	Causal sets	334
29.3	Quantum causal sets	336
29.4	Discrete time and the evolving observer	336
<b>30</b>	<b>Concluding remarks</b>	<b>341</b>
<b>Appendix A</b>	<b>Coherent states</b>	<b>343</b>
<b>Appendix B</b>	<b>The time-dependent oscillator</b>	<b>345</b>
<b>Appendix C</b>	<b>Quaternions</b>	<b>347</b>
<b>Appendix D</b>	<b>Quantum registers</b>	<b>348</b>
<i>References</i>		353
<i>Index</i>		361

## Preface

Long ago, great minds speculated on the nature of time. The following question was asked: could time be divided into ever smaller and smaller pieces, just like a length of wood? We know this for a historical fact, because some of Zeno's paradoxes have survived the ravages of time and these paradoxes discuss precisely this question.

Contrary to what might be believed, interest in Zeno's paradoxes has not been extinguished by the rigours of modern mathematics, although we are taught that it has. Yes, the paradox of Achilles and the tortoise can be explained away in terms of a convergent infinite sum. But the concept of an infinitesimal has not been killed off: far from it, for mathematicians have developed a rigorous, consistent mathematical framework called non-standard analysis that allows for such things.

What I believe this debate about time highlights is how conditioned humans can be. We learn from an early age to think in certain terms and, if we are not careful, we end up regarding them as the only possible framework for our thoughts. So it is with time, which has been regarded as continuous throughout the history of mathematics and physics. It is hard to imagine any physical theory without the concept of a time derivative, and that requires continuity in time.

However, it is the obligation of theorists not only to explore current theories to their natural horizons, but to look beyond those horizons and to step outside of them if that is possible. That's really what theorists are paid for, not for the confirmation of established paradigms. I started to be concerned about standard physics when I first encountered wavefunction renormalization, that notorious method of dealing with the divergences of quantum field theory. Now, many years later, I can see that this concern was a portent of what was to come, for a very large quantity can be regarded as the reciprocal of a very small quantity. Very large energies and momenta are related to very small timescales and intervals of space, as I shall discuss.

Can we resolve these problems? Is it possible to understand Zeno's paradoxes about the vanishingly small and understand the divergences in quantum field theory within the same framework?

I think the answer is *possibly*, but it will require a deeper examination of the role of the observer. The observer has long taken a back seat in scientific theory,

because the focus in science has generally been on the systems under observation. It is my belief that the balance has to be redressed, particularly when it comes to time.

This book is not about approximations to continuous time models but an exploration of discrete time as a model in its own right. I am not interested in finding good discrete time approximations to continuous time equations or their solutions. So do not look here for advice about the latest and best convergent lattice discretizations in fluid mechanics, non-abelian gauge theory or gravitation. There are plenty of sources on those topics. I am exploring the following question: what would be the consequences of the conjecture that time is really discrete?

This book will necessarily be centred on my own experiences: what I have read, what I have written, and what has come to me through talking to others. So I will inevitably have missed some important topics and papers written by others, for which I apologize profusely in advance.

A preface is generally the place where an author expresses their unbounded gratitude to others. I do so now. I am indebted to all my teachers and lecturers, colleagues and students down the years who have given me far more of value than I have given them. In particular, I benefited from the wisdom and inspiration of Professors Nicholas Kemmer, Julian Schwinger and Peter Landshoff at various points in my career. I am indebted also to Andy Walker and Anne Lomax, and to Volodia and Rummy Nikolaev.

I thank also all my research students who took the risk of working with me on discrete time: Keith Norton, Jon Eakins, Jason Ridgway-Taylor and Fernando Aguayo.

My deepest thanks go to all members of my family, past and present: my wife Małgorzata and daughter Joanna for their endless patience and support, and to my parents for the priceless values they gave me.

George Jaroszkiewicz,  
Walton on the Wolds