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Introduction

Purpose This book is an introduction to some of the basic concepts of topology, especially of *non-Hausdorff* topology. I will of course explain what it means (Definition 4.1.12). The important point is that traditional topology textbooks assume the Hausdorff separation condition from the very start, and contain very little information on non-Hausdorff spaces. But the latter are important already in algebraic geometry, and crucial in fields such as domain theory.

Conversely, domain theory (Abramsky and Jung, 1994; Gierz *et al.*, 2003), which arose from logic and computer science, started as an outgrowth of theories of order. Progress in this domain rapidly required a lot of material on (non-Hausdorff) topologies.

After about 40 years of domain theory, one is forced to recognize that topology and domain theory have been beneficial to each other. I've already mentioned what domain theory owes to topology. Conversely, in several respects, domain theory, in a broad sense, is *topology done right*.

This book is an introduction to both fields, dealt with as one. This seems to fill a gap in the literature, while bringing them forth in a refreshing perspective.

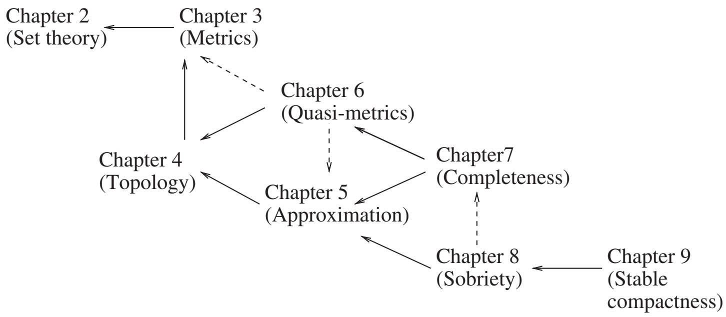
Secondary purpose This book is self-contained. My main interest, though, as an author, was to produce a unique reference for the kind of results in topology and domain theory that I needed in research I started in 2004, on semantic models of mixed non-deterministic and probabilistic choice. The goal quickly grew out of proportion, and will therefore occupy several volumes. The current book can be seen as the preliminaries for other books on these other topics. Some colleagues of mine, starting with Professor Alain Finkel, have stressed that these preliminaries are worthy of interest, independently of any specific application.

Teaching This book is meant to be used as a reference. My hope is that it should be useful to researchers and to people who are curious to read about a modern view of point-set topology. It did not arise as lecture notes, and I don't think it can be used as such directly. If you plan on using this book as a basis for lectures, you should extract a few selected topics. Think of yourself as a script writer and this book as a novel, and pretend that your job is to produce a shorter script for a feature-length movie, and skip the less important subplots.

Exercises There are many exercises, spread over the whole text. Some of them are meant for training, i.e., to help you understand notions better, to make you more comfortable with definitions and theorems that we have just seen. Some others are there to help you understand notions in a deeper way, or to go further. This is traditional in mathematical textbooks. It is therefore profitable to read all exercises: those of the second kind in particular state additional theorems, which you can prove for yourself (sometimes with the help of hints), but do not need to. It will happen that not only solutions of exercises but also some proofs of theorems will depend on results we shall have seen in previous exercises. There is no pressure on you to actually do any exercise, and you can decide to take them as a mere source of additional information.

What is not covered Topology is an extremely rich topic, and I could not cover all subtopics in a book of reasonable size. (Notwithstanding the fact that I certainly do not know everything in topology.) I decided to make a selection among those subtopics that pleased me most. Some other topics were necessarily left out, despite them being equally interesting. For example, algebraic topology will not be touched upon at all. Uniform and quasi-uniform spaces, bitopological spaces, Lindelöf spaces, Souslin spaces, and topological group theory were left out as well. Topological convexity, topological measure theory, hyperspaces, and powerdomains will be treated in further volumes.

Dependencies Reading $u \leftarrow v$ as "v depends on u," the structure of the book is as follows. Dashed arrows mark weaker dependencies.



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Elements of set theory

We recapitulate the axiomatic foundations on which this book is based in Section 2.1. We then recall a few points about finiteness and countability in Section 2.2 and some basics of order theory in Section 2.3, and discuss the Axiom of Choice and some of its consequences in Section 2.4. These points will be needed often in the rest of this book. If you prefer to read about topology right away, and feel confident enough, please proceed directly to Chapter 3.

2.1 Foundations

We shall rest on ordinary set theory. While the latter has been synonymous with Zermelo–Fraenkel (ZF) set theory with the Axiom of Choice (ZFC) for some time, we shall use von Neumann–Gödel–Bernays (VBG) set theory instead (Mendelson, 1997).

There is not much difference between these theories: VBG is a conservative extension of ZFC. That VBG is an extension means that any theorem of ZFC is also a theorem of VBG. That it is conservative means that any theorem of VBG that one can express in the language of ZFC is also provable in ZFC.

The main difference between VBG and ZFC is that the former allows one to talk about collections that are too big to be sets. This is required, in all rigor, in the definition of (big) graphs and categories of Section 4.12. VBG allows us to talk about, say, the collection V of all sets, although V cannot itself be a set. This is the essence of Russell's paradox: assume there were a set V of all sets. Then $A = \{x \in V \mid x \notin x\}$ is a set. The rather mind-boggling argument is that, first, $A \notin A$ since if A were in A, then A would be an x such that $x \notin x$ by definition of A. Since $A \notin A$, A is an x such that $x \notin A$, so A is in A, a contradiction.

VBG avoids this pitfall by not recognizing V as a set, but as something of a different nature called a collection (or a *class*). Here is an informal description of the axioms of VBG set theory.

VBG set theory is a theory of *collections*. The formulae of VBG are those obtained using the language of first-order logic, i.e., using true ($\top$), false ($\bot$), logical and ($\wedge$), or ($\vee$), negation ($\neg$), implication ($\Rightarrow$), equivalence ($\Leftrightarrow$), and universal ($\forall$) and existential ($\exists$) quantification, on a language whose predicate symbols are $\in$ ("belongs to," "is in," "is an element of"), $=$ (equality), and m ("being small"). The small collections, i.e., the collections A such that $m(A)$ holds, are called *sets*. Collections must obey the following axioms:

- (Elements are small) Any element of a collection is small: $\forall x, A \cdot x \in A \Rightarrow m(x)$.
- (Extensionality) Any two collections with the same elements are equal: $\forall A, B \cdot (\forall x \cdot x \in A \Leftrightarrow x \in B) \Rightarrow A = B$.
- (Comprehension) Say that a formula F is a set formula iff all the quantifications in F are over sets, not collections. That is, the syntax of set formulae is $F, G, \ldots ::= \top \mid \bot \mid m(x) \mid x \in y \mid x = y \mid F \wedge G \mid F \vee G \mid \neg F \mid F \Rightarrow G \mid F \Leftrightarrow G \mid \forall x \cdot m(x) \Rightarrow F \mid \exists x \cdot m(x) \wedge F$. The comprehension schema states that, for each set formula $F(x, \vec{y})$ (where we make explicit the set of collections variables x, $\vec{y} = y_1, \ldots, y_n$ that it may depend on), there is a collection $\{x \mid F(x, \vec{y})\}$ of all elements x such that $F(x, \vec{y})$ holds: $\forall \vec{y} \cdot \exists A \cdot \forall x \cdot x \in A \Leftrightarrow F(x, \vec{y})$.
- (Empty set) the empty collection $\emptyset = \{x \mid \bot\}$ is a set: $m(\emptyset)$.
- (Pairing) Given any two sets x, y, their pair $\{x, y\} = \{z \mid z = x \vee z = y\}$ is a set: $m(\{x, y\})$. The ordered pair (x, y) is encoded as $\{\{x, x\}, \{x, y\}\}$. It follows that $\{x\} = \{x, x\}$ is a set as well.
- (Union) Given any set x, the collection $\bigcup x = \bigcup_{y \in x} y = \{z \mid \exists y \cdot m(y) \wedge y \in x\}$ is a set; i.e., $m(\{z \mid \exists y \cdot m(y) \wedge y \in x\})$. We write $x \cup y$ for $\bigcup\{x, y\}$.
- (Powerset) Given any set x, the collection $\mathbb{P}(x)$ of all subsets of x is a set, i.e., $m(\{z \mid z \subseteq x\})$, where we write $z \subseteq x$ for $\forall y \cdot m(y) \Rightarrow (y \in z \Rightarrow y \in x)$, or equivalently $\forall y \cdot y \in z \Rightarrow y \in x$. For every set A, $\mathbb{P}(A)$ is called the *powerset* of A.
- (Infinity) Let $0 = \emptyset$, $x + 1 = x \cup \{x\}$. There is a collection $\mathbb{N}$ containing 0 and such that $x \in \mathbb{N}$ implies $x + 1 \in \mathbb{N}$. One may even define the smallest such collection, $\mathbb{N}$, as $\{x \mid \forall N \cdot m(N) \Rightarrow (0 \in N \wedge (\forall z \cdot m(z) \Rightarrow (z \in N \Rightarrow z + 1 \in N)) \Rightarrow x \in N)\}$. The axiom of infinity states that $\mathbb{N}$ is small: $m(\mathbb{N})$. $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \ldots\}$.

- (Foundation) There is no infinite chain $\cdots \in x_k \in \cdots \in x_2 \in x_1$. This is usually formulated in the more cryptic, but equivalent form: Every non-empty class is disjoint from one of its elements. One may as well axiomatize this by an induction axiom, stating that we may induct along the relation $\in$: $\forall A \cdot (\forall x \cdot m(x) \wedge (\forall y \cdot y \in x \Rightarrow y \in A) \Rightarrow x \in A) \Rightarrow \forall x \cdot m(x) \Rightarrow x \in A$, i.e., any class A that contains every set x whenever it contains all smaller sets (i.e., all sets y such that $y \in x$) must in fact contain all sets.
- (Replacement) The image of a set under a function is a set. Functions f are encoded as their graphs, i.e., as the class of all pairs $(x, f(x))$: so a function is any class f such that $\forall x, y, z \cdot (x, y) \in f \wedge (x, z) \in f \Rightarrow y = z$. Replacement states that, for every function f and every set A, its *image* $f[A] = \{y \mid \exists x \cdot m(x) \wedge x \in A \wedge (x, y) \in f\}$ is small.
- (Axiom of Choice) Given any function f, call the *domain* of f the set of elements x such that $f(x)$ is defined, i.e., the set of elements x such that, for some y $(= f(x))$, (x, y) is in f. The Axiom of Choice states that, for every function F such that, for every x in its domain, $F(x)$ is a non-empty set, there is a function f whose domain is the same as F, and such that $f(x) \in F(x)$ for every x in the domain of F. In other words, f singles out one element $f(x)$ from each set $F(x)$. We discuss this axiom in Section 2.4.

We use standard abbreviations throughout; e.g., a set A *intersects*, or *meets*, a set B if and only if $A \cap B$ is non-empty, i.e., if and only if $A \cap B \neq \emptyset$.

2.2 Finiteness, countability

A set A is *finite* iff one can write it $\{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$. We can assume that $x_1, \ldots, x_n$ are pairwise distinct. A formal definition may be: A is finite iff there is a bijection from A to some subset of the form $\downarrow n = \{m \in \mathbb{N} \mid m \leqslant n\}$, $n \in \mathbb{N}$.

A set is *infinite* iff it is not finite.

A set A is *countable* iff there is a bijection from A and some (arbitrary) subset of $\mathbb{N}$. It is equivalent to say that A is finite or *countably infinite*, where A is countably infinite iff there is a bijection from A to the whole of $\mathbb{N}$.

Every non-empty countable set A can be written $\{x_i \mid i \in \mathbb{N}\}$, i.e., there is a surjective map $i \mapsto x_i$ from $\mathbb{N}$ to A. (This is not true if A is empty.)

Every subset of a finite set is finite, and similarly every subset of a countable set is countable. In particular, any non-empty intersection of countable sets is countable.

We observe that $\mathbb{N} \times \mathbb{N}$ is countable. One possible bijection $m, n \mapsto \ulcorner m, n \urcorner$ is defined as follows. We can write any natural number in base 2, as an infinite word $\ldots a_i a_{i-1} \ldots a_2 a_1 a_0$, where each a_i is in $\{0, 1\}$, and all but finitely many equal 0: this word represents the natural number $\sum_{i=0}^{+\infty} a_i 2^i$. Now, when m is written in base 2 as $\ldots a_i a_{i-1} \ldots a_2 a_1 a_0$ and n as $\ldots b_i b_{i-1} \ldots b_2 b_1 b_0$, define $\ulcorner m, n \urcorner = \ldots a_i b_i a_{i-1} b_{i-1} \ldots a_2 b_2 a_1 b_1 a_0 b_0$.

It follows that the product $A \times B = \{(x, y) \mid x \in A, y \in B\}$ of two countable sets is countable.

Every countable union of countable sets is countable; i.e., (for non-empty sets) if A_k is the countable set $\{x_{ki} \mid i \in \mathbb{N}\}$ for each $k \in \mathbb{N}$, then $\bigcup_{k \in \mathbb{N}} A_k$ is the set $\{x_{ki} \mid \ulcorner k, i \urcorner \in \mathbb{N}\}$, which is countable.

We deduce that the set $\mathbb{Z}$ of all integers, negative, zero, or positive, is countable: indeed $\mathbb{Z}$ is in bijection with the union of $\mathbb{N}$ with $\{-n-1 \mid n \in \mathbb{N}\}$.

It also follows that the set $\mathbb{Q}$ of rational numbers is countable. Indeed, consider the set A of pairs $(m, n) \in \mathbb{Z} \times \mathbb{N}$ such that m and $n+1$ have no common divisor other than 1. A is countable, as a subset of the countable set $\mathbb{Z} \times \mathbb{N}$, and there is a bijection from A to $\mathbb{Q}$, which sends (m, n) to $\frac{m}{n+1}$.

The *finite powerset* $\mathbb{P}_{\text{fin}}(A)$ of a set A is the set of all finite subsets of A. When A is countable, $\mathbb{P}_{\text{fin}}(A)$ is countable, too. This is obvious when A is empty; otherwise it is enough to show that $\mathbb{P}_{\text{fin}}(\mathbb{N})$ is countable: the required bijection $E \in \mathbb{P}_{\text{fin}}(\mathbb{N}) \mapsto \ulcorner E \urcorner \in \mathbb{N}$ can be defined, for example, by $\ulcorner E \urcorner = \sum_{i \in E} 2^i$; its inverse maps every number m written in base 2 as $\ldots a_i a_{i-1} \ldots a_2 a_1 a_0$ to the finite set of all indices i such that $a_i = 1$.

However, the *powerset* $\mathbb{P}(A)$ of *all* subsets of A is not countable in general, even when A is countable. To wit, $\mathbb{P}(\mathbb{N})$ is not countable. This would indeed imply the existence of a surjective map from $\mathbb{N}$ to $\mathbb{P}(\mathbb{N})$, contradicting *Cantor's Theorem*:

Theorem 2.2.1 (Cantor) *For any set A, there is no surjective map from A to $\mathbb{P}(A)$.*

Proof Assume there was one, say f. Let $B = \{x \in A \mid x \notin f(x)\}$. Since f is surjective, there is an $x \in A$ such that $B = f(x)$. If $x \in B$, i.e., if $x \in f(x)$, by definition x is not in B: contradiction. So x is not in B. This implies that $x \notin f(x)$ since $B = f(x)$, so $x \in B$ by definition of B, again a contradiction. □

We call *uncountable* any set that is not countable.

The set $\{0, 1\}^\omega$ of all infinite words on the alphabet $\{0, 1\}$ is not countable either. An *infinite word* on an alphabet Σ is just a sequence of letters in Σ, i.e., a map from $\mathbb{N}$ to Σ. The map sending $w \in \{0, 1\}^\omega$ to the set $\{i \in \mathbb{N} \mid w(i) = 1\}$ is a bijection between $\{0, 1\}^\omega$ and $\mathbb{P}(\mathbb{N})$, so $\{0, 1\}^\omega$ cannot be countable.

It follows that the interval $[0, 1]$ of $\mathbb{R}$ is not countable either. Indeed, define the following function: for each $x \in [0, 1)$, write x in base 3 as $\sum_{k=0}^{+\infty} a_k/3^{k+1}$; if every a_k is in $\{0, 2\}$, then map x to the infinite word $\frac{a_0}{2}\frac{a_1}{2}\ldots\frac{a_k}{2}\ldots$; otherwise map x to an arbitrary word; finally, map 1 to the all one word $11\ldots1\ldots$ This is clearly surjective. If $[0, 1]$ were countable, there would also be a surjection from $\mathbb{N}$ to $[0, 1]$, which, composed with the above surjection, would yield a surjection from $\mathbb{N}$ to $\mathbb{P}(\mathbb{N})$, contradicting Cantor's Theorem.

From this, we deduce that $\mathbb{R}$ itself is not countable either. In fact, no subset A of $\mathbb{R}$ that contains a non-empty interval $[a-\epsilon, a+\epsilon]$, $\epsilon > 0$, can be countable. Indeed, this interval is in bijection with $[0, 1]$, by the map $t \mapsto (t-a+\epsilon)/(2\epsilon)$.

2.3 Order theory

2.3.1 Orderings, quasi-orderings

A *binary relation* R on X is any subset of $X \times X$. We write $x \mathrel{R} y$ instead of $(x, y) \in R$. R is *reflexive* iff $x \mathrel{R} x$ for every $x \in X$. R is *transitive* iff whenever $x \mathrel{R} y$ and $y \mathrel{R} z$, then $x \mathrel{R} z$. A reflexive and transitive relation is called a *quasi-ordering*, and is usually written $\leqslant$. A *quasi-ordered set* is any set equipped with a quasi-ordering. When $x \leqslant y$, we say that x is *below* y, or that y is *above* x.

R is *antisymmetric* iff whenever $x \mathrel{R} y$ and $y \mathrel{R} x$, then $x = y$. A *partial ordering*, or *ordering* for short, on X is any antisymmetric quasi-ordering on X. A set with an ordering is called a *partially ordered set* or a *poset*.

A binary relation R is *symmetric* iff whenever $x \mathrel{R} y$, then $y \mathrel{R} x$. An *equivalence relation* is any reflexive, symmetric, and transitive relation. We shall usually write $\equiv$ for equivalence relations. The *equivalence class* $q_\equiv(x)$ of $x \in X$ is the set of all $y \in X$ such that $x \equiv y$. The *quotient* $X/{\equiv}$ is the set of all equivalence classes of X.

Given any quasi-ordering $\leqslant$ on X, we can define an equivalence relation $\equiv$ by $x \equiv y$ iff $x \leqslant y$ and $y \leqslant x$. If $\leqslant$ is an ordering, then $\equiv$ is just the equality relation $=$. Otherwise, define $q_\equiv(x) \leqslant q_\equiv(y)$ iff $x \leqslant y$, and observe that this is well defined, i.e., it does not depend on the exact elements x and y that one picks from the equivalence classes $q_\equiv(x)$ and $q_\equiv(y)$. Then $\leqslant$ is not just a quasi-ordering, but is an ordering on $X/{\equiv}$. So, up to quotients, there is not much difference between quasi-ordered sets and posets.

We also write $<$ for the *strict part* of the quasi-ordering $\leqslant$, viz., $x < y$ iff $x \leqslant y$ and $x \not\equiv y$, iff $x \leqslant y$ and $y \not\leqslant x$.

Write $y \geqslant x$ iff $x \leqslant y$, and call $\geqslant$ the *opposite* of $\leqslant$. For every poset X, the set of all elements of X ordered by $\geqslant$ is the *opposite* of X, and is written X^{op}. We write $>$ for the strict part of $\geqslant$.

In any quasi-ordered set X, we say that a subset A is *upward closed* iff whenever $x \in A$ and $x \leqslant y$, then $y \in A$: going up from an element of A, we stay in A. The *upward closure* $\uparrow_X A$ of A in X is the set $\{y \in X \mid \exists x \in A \cdot x \leqslant y\}$ of all elements above some element of A. For short, we write $\uparrow A$ when X is clear from context, and $\uparrow x$ for $\uparrow\{x\}$ when $x \in X$. Note that A is upward closed iff $A = \uparrow A$. Conversely, we say that A is *downward closed* iff every element below some element of A is again in A. The *downward closure* $\downarrow_X A$, or $\downarrow A$ when X is clear, equals $\{y \in X \mid \exists x \in A \cdot y \leqslant x\}$. For short, we shall write $\downarrow x$ for $\downarrow\{x\}$ when $x \in X$.

2.3.2 Upper bounds, lower bounds

For any subset A of a poset X, we say that $x \in A$ is a *least element* of A iff $A \subseteq \uparrow x$, i.e., every element of A is above x. The least element, if it exists, is unique. This should not be confused with the notion of *minimal element*: $x \in A$ is *minimal* in A iff no element of A is strictly below x, i.e., iff whenever $y \leqslant x$ and $y \in A$, then $y = x$. The least element of A, if it exists, is minimal, but minimal elements may fail to be least. For example, in $\mathbb{N} \times \mathbb{N}$ with the product ordering, defined by $(m, n) \leqslant (m', n')$ iff $m \leqslant m'$ and $n \leqslant n'$, the set $\uparrow\{(1, 2), (3, 1)\}$ has two minimal elements, $(1, 2)$ and $(3, 1)$, but no least element.

The *greatest* element and the notion of *maximal* elements are defined similarly, using $\geqslant$ instead of $\leqslant$.

An *upper bound* x of a subset A of a poset X is an element such that $y \leqslant x$ for every $y \in A$. That is, an upper bound of A sits above every element of A. In $\mathbb{R} \times \mathbb{R}$, for example, $(3, 1)$, $(5, 5)$, and $(6, 1)$ are some of the upper bounds of the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x < 3, y < 1\}$.

The *least upper bound* of a subset A of a poset X, if it exists, is the least element of the set of all upper bounds of A in X. For short, we shall also call it the *supremum* of A, and write it $\sup A$. For example, the subset $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x < 3, y < 1\}$ of $\mathbb{R} \times \mathbb{R}$ admits $(3, 1)$ as least upper bound. When A is a family $(x_i)_{i \in I}$, we also write $\sup_{i \in I} x_i$ instead of $\sup A$.

If $\sup A$ exists and is in A, then $\sup A$ is necessarily the greatest element of A. But $\sup A$ may fail to be in A, as in the example just given.

The notions of *lower bound*, *greatest lower bound*, a.k.a. *infimum* $\inf A$ of a subset A of X, are defined similarly, replacing $\leqslant$ by its opposite $\geqslant$.

A *complete lattice* is a poset X such that every subset A of X has a least upper bound $\sup A$ and a greatest lower bound $\inf A$. It is equivalent to require the existence of least upper bounds only, since then the greatest lower bounds $\inf A$ exist as well, as least upper bounds of the set of lower bounds of A.

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Similarly, any poset in which every subset has a greatest lower bound is also a complete lattice.

Every complete lattice has a least element $\bot$ (also called *bottom*), obtained as the least upper bound $\sup \emptyset$ of the empty family, and a greatest element $\top$ (also called *top*), obtained as $\inf \emptyset$.

For any set X, $\mathbb{P}(X)$ with the inclusion ordering $\subseteq$ is a complete lattice: the least upper bound of a family of subsets of X is their union, and the greatest lower bound is their intersection.

There are various weaker notions. For example, an *inf-semi-lattice* is a poset in which any two elements x, y have a greatest lower bound $x \wedge y$, and therefore any non-empty finite set of elements has a greatest lower bound. Symmetrically, a *sup-semi-lattice* is a poset in which any two elements x, y have a least upper bound $x \vee y$, and therefore any non-empty finite set of elements has a least upper bound.

We do not require inf-semi-lattices to have greatest lower bounds of all finite subsets, only the non-empty ones. That is, we do not require them to have a largest element $\top$. Similarly, we do not require sup-semi-lattices to have a least element $\bot$. Those that have one have least upper bounds of all finite subsets, and are called *pointed*.

A *lattice* is any poset that is both an inf-semi-lattice and a sup-semi-lattice. Those lattices that have a least element $\bot$ and a largest element $\top$ are called *bounded*.

2.3.3 Fixed point theorems

A map f from a poset X to a quasi-ordered set Y is *monotonic* iff, for every $x, x' \in X$ with $x \leqslant x'$, $f(x) \leqslant f(x')$. When X is a poset, an *order embedding* $f : X \to Y$ is a monotonic map such that, additionally, whenever $f(x) \leqslant f(x')$, then $x \leqslant x'$. In particular, every order embedding is injective, but there are injective monotonic maps that are not order embeddings, e.g., the identity map from $\{0, 1\}$ with equality as ordering to $\{0, 1\}$ with the ordering $0 \leqslant 1$. An *order isomorphism* $f : X \to Y$ is a bijective, monotonic map, whose inverse is also monotonic. Every order isomorphism is an order embedding, and every order embedding $f : X \to Y$ defines an order isomorphism from X to the image of X by f in Y, with the ordering induced by that of Y.

Two elements x and y of a quasi-ordered set X are *incomparable* iff $x \not\leqslant y$ and $y \not\leqslant x$. For example, $(5, 5)$ and $(6, 1)$ are incomparable in $\mathbb{R} \times \mathbb{R}$.

A subset A of a poset X is *totally ordered* iff it has no pair of incomparable elements, i.e., iff, for every pair of elements x, y of A, $x \leqslant y$ or $y \leqslant x$. The ordering $\leqslant$ itself is called *total*, or *linear*, iff X is totally ordered by $\leqslant$. A *chain* in X is any totally ordered, non-empty subset of X.