CONFORMAL METHODS IN GENERAL RELATIVITY

This book is a systematic exposition of conformal methods and how they can be used to study the global properties of solutions to the equations of Einstein's theory of gravity. It shows that combining these ideas with techniques of the theory of partial differential equations can elucidate the stability of the basic solutions of the theory. Introducing the differential geometric, spinorial and PDE background required to gain a deep understanding of conformal methods, this text provides an accessible account of key results in mathematical relativity over the last 30 years, including the stability of de Sitter and Minkowski spacetimes.

For graduate students and researchers, this self-contained account includes useful visual models to help the reader grasp abstract concepts and a list of further reading, making this the perfect reference companion on the topic.

Juan A. Valiente Kroon is a Reader in Applied Mathematics at Queen Mary University of London. He was a Lise Meitner fellow of the Austrian Science Fund (FWF), an Engineering and Physical Sciences (EPSRC) Advanced Research fellow and he specialises in various aspects of mathematical general relativity.

CAMBRIDGE MONOGRAPHS ON MATHEMATICAL PHYSICS

General Editors: P. V. Landshoff, D. R. Nelson, S. Weinberg

S. J. Aarseth *Gravitational N-Body Simulations: Tools and Algorithms*†
J. Ambjørn, B. Durhuus and T. Jonsson *Quantum Geometry: A Statistical Field Theory Approach*†
A. M. Anile *Relativistic Fluids and Magneto-fluids: With Applications in Astrophysics and Plasma Physics*
J. A. de Azcárraga and J. M. Izquierdo *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics*†
O. Babelon, D. Bernard and M. Talon *Introduction to Classical Integrable Systems*†
F. Bastianelli and P. van Nieuwenhuizen *Path Integrals and Anomalies in Curved Space*†
D. Baumann and L. McAllister *Inflation and String Theory*
V. Belinski and E. Verdaguer *Gravitational Solitons*†
J. Bernstein *Kinetic Theory in the Expanding Universe*†
G. F. Bertsch and R. A. Broglia *Oscillations in Finite Quantum Systems*†
N. D. Birrell and P. C. W. Davies *Quantum Fields in Curved Space*†
K. Bolejko, A. Krasiński, C. Hellaby and M-N. Célérier *Structures in the Universe by Exact Methods: Formation, Evolution, Interactions*
D. M. Brink *Semi-Classical Methods for Nucleus-Nucleus Scattering*†
M. Burgess *Classical Covariant Fields*†
E. A. Calzetta and B.-L. B. Hu *Nonequilibrium Quantum Field Theory*
S. Carlip *Quantum Gravity in 2+1 Dimensions*†
P. Cartier and C. DeWitt-Morette *Functional Integration: Action and Symmetries*†
J. C. Collins *Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion*†
P. D. B. Collins *An Introduction to Regge Theory and High Energy Physics*†
M. Creutz *Quarks, Gluons and Lattices*†
P. D. D'Eath *Supersymmetric Quantum Cosmology*†
J. Dereziński and C. Gérard *Mathematics of Quantization and Quantum Fields*
F. de Felice and D. Bini *Classical Measurements in Curved Space-Times*
F. de Felice and C. J. S Clarke *Relativity on Curved Manifolds*†
B. DeWitt *Supermanifolds, 2nd edition*†
P. G. O. Freund *Introduction to Supersymmetry*†
F. G. Friedlander *The Wave Equation on a Curved Space-Time*†
J. L. Friedman and N. Stergioulas *Rotating Relativistic Stars*
Y. Frishman and J. Sonnenschein *Non-Perturbative Field Theory: From Two Dimensional Conformal Field Theory to QCD in Four Dimensions*
J. A. Fuchs *Affine Lie Algebras and Quantum Groups: An Introduction, with Applications in Conformal Field Theory*†
J. Fuchs and C. Schweigert *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists*†
Y. Fujii and K. Maeda *The Scalar-Tensor Theory of Gravitation*†
J. A. H. Futterman, F. A. Handler, R. A. Matzner *Scattering from Black Holes*†
A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev *Harmonic Superspace*†
R. Gambini and J. Pullin *Loops, Knots, Gauge Theories and Quantum Gravity*†
T. Gannon *Moonshine beyond the Monster: The Bridge Connecting Algebra, Modular Forms and Physics*†
M. Göckeler and T. Schücker *Differential Geometry, Gauge Theories and Gravity*†
C. Gómez, M. Ruiz-Altaba and G. Sierra *Quantum Groups in Two-Dimensional Physics*†
M. B. Green, J. H. Schwarz and E. Witten *Superstring Theory Volume 1: Introduction*
M. B. Green, J. H. Schwarz and E. Witten *Superstring Theory Volume 2: Loop Amplitudes, Anomalies and Phenomenology*
V. N. Gribov *The Theory of Complex Angular Momenta: Gribov Lectures on Theoretical Physics*†
J. B. Griffiths and J. Podolský *Exact Space-Times in Einstein's General Relativity*†
S. W. Hawking and G. F. R. Ellis *The Large Scale Structure of Space-Time*†
F. Iachello and A. Arima *The Interacting Boson Model*†
F. Iachello and P. van Isacker *The Interacting Boson-Fermion Model*†
C. Itzykson and J. M. Drouffe *Statistical Field Theory Volume 1: From Brownian Motion to Renormalization and Lattice Gauge Theory*†
C. Itzykson and J. M. Drouffe *Statistical Field Theory Volume 2: Strong Coupling, Monte Carlo Methods, Conformal Field Theory and Random Systems*†
G. Jaroszkiewicz *Principles of Discrete Time Mechanics*
C. V. Johnson *D-Branes*†
P. S. Joshi *Gravitational Collapse and Spacetime Singularities*†

J. I. Kapusta and C. Gale *Finite-Temperature Field Theory: Principles and Applications, 2nd edition*†
V. E. Korepin, N. M. Bogoliubov and A. G. Izergin *Quantum Inverse Scattering Method and Correlation Functions*†
M. Le Bellac *Thermal Field Theory*†
Y. Makeenko *Methods of Contemporary Gauge Theory*†
N. Manton and P. Sutcliffe *Topological Solitons*†
N. H. March *Liquid Metals: Concepts and Theory*†
I. Montvay and G. Münster *Quantum Fields on a Lattice*†
L. O'Raifeartaigh *Group Structure of Gauge Theories*†
T. Ortín *Gravity and Strings, 2nd edition*
A. M. Ozorio de Almeida *Hamiltonian Systems: Chaos and Quantization*†
L. Parker and D. Toms *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*
R. Penrose and W. Rindler *Spinors and Space-Time Volume 1: Two-Spinor Calculus and Relativistic Fields*†
R. Penrose and W. Rindler *Spinors and Space-Time Volume 2: Spinor and Twistor Methods in Space-Time Geometry*†
S. Pokorski *Gauge Field Theories, 2nd edition*†
J. Polchinski *String Theory Volume 1: An Introduction to the Bosonic String*†
J. Polchinski *String Theory Volume 2: Superstring Theory and Beyond*†
J. C. Polkinghorne *Models of High Energy Processes*†
V. N. Popov *Functional Integrals and Collective Excitations*†
L. V. Prokhorov and S. V. Shabanov *Hamiltonian Mechanics of Gauge Systems*
A. Recknagel and V. Schiomerus *Boundary Conformal Field Theory and the Worldsheet Approach to D-Branes*
R. J. Rivers *Path Integral Methods in Quantum Field Theory*†
R. G. Roberts *The Structure of the Proton: Deep Inelastic Scattering*†
C. Rovelli *Quantum Gravity*†
W. C. Saslaw *Gravitational Physics of Stellar and Galactic Systems*†
R. N. Sen *Causality, Measurement Theory and the Differentiable Structure of Space-Time*
M. Shifman and A. Yung *Supersymmetric Solitons*
H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt *Exact Solutions of Einstein's Field Equations, 2nd edition*†
J. Stewart *Advanced General Relativity*†
J. C. Taylor *Gauge Theories of Weak Interactions*†
T. Thiemann *Modern Canonical Quantum General Relativity*†
D. J. Toms *The Schwinger Action Principle and Effective Action*†
A. Vilenkin and E. P. S. Shellard *Cosmic Strings and Other Topological Defects*†
R. S. Ward and R. O. Wells, Jr *Twistor Geometry and Field Theory*†
E. J. Weinberg *Classical Solutions in Quantum Field Theory: Solitons and Instantons in High Energy Physics*
J. R. Wilson and G. J. Mathews *Relativistic Numerical Hydrodynamics*†
T. Ortín *Gravity and Strings*
D. Baumann and L. McAllister *Inflation and String Theory*
S. Raychaudhuri and K. Sridhar *Particle Physics of Brane Worlds and Extra Dimensions*

† Available in paperback

Conformal Methods in General Relativity

JUAN A. VALIENTE KROON
Queen Mary University of London

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107033894

First published 2016

Printed in the United Kingdom by TJ International Ltd. Padstow Cornwall

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication data
Names: Valiente Kroon, Juan A., 1972– author.
Title: Conformal methods in general relativity / Juan A. Valiente Kroon (Queen Mary, University of London).
Description: Cambridge, United Kingdom ; New York, NY : Cambridge University Press, 2016. |
Includes bibliographical references and index.
Identifiers: LCCN 2016015466 | ISBN 9781107033894 (Hardback ; alk. paper) | ISBN 1107033896 (Hardback ; alk. paper)
Subjects: LCSH: General relativity (Physics)–Mathematics. | Geometry, Differential. | Conformal mapping. | Conformal geometry. | Einstein field equations–Numerical solutions. | Space and time.
Classification: LCC QC173.6 .V35 2016 | DDC 530.11–dc23
LC record available at https://lccn.loc.gov/2016015466

ISBN 978-1-107-03389-4 Hardback

Cambridge University Press
978-1-107-03389-4 - Conformal Methods in General Relativity
Juan A. Valiente Kroon
Frontmatter
More information

ἀγεωμέτρητος μηδεὶς εἰςίτω (Let no one untrained in geometry enter)

– *Epigram at the Academy of Plato*

Contents

Preface *page* xv
Acknowledgements xviii
List of Symbols xx

1 Introduction 1
1.1 On the Einstein field equations 1
1.2 Exact solutions 3
1.3 The Cauchy problem in general relativity 4
1.4 Conformal geometry and general relativity 8
1.5 Existence of asymptotically simple spacetimes 15
1.6 Perspectives 21
1.7 Structure of this book 21

Part I Geometric tools **25**

2 Differential geometry 27
2.1 Manifolds 27
2.2 Vectors and tensors on a manifold 30
2.3 Maps between manifolds 36
2.4 Connections, torsion and curvature 38
2.5 Metric tensors 44
2.6 Frame formalisms 51
2.7 Congruences and submanifolds 53
2.8 Further reading 63

3 Spacetime spinors 64
3.1 Algebra of 2-spinors 65
3.2 Calculus of spacetime spinors 81
3.3 Global considerations 90
3.4 Further reading 91
Appendix: the Newman-Penrose formalism 91

4 Space spinors 94
4.1 Hermitian inner products and 2-spinors 94
4.2 The space spinor formalism 97
4.3 Calculus of space spinors 105
4.4 Further reading 111

5 Conformal geometry 112
5.1 Basic concepts of conformal geometry 112
5.2 Conformal transformation formulae 114
5.3 Weyl connections 119
5.4 Spinorial expressions 122
5.5 Conformal geodesics 126
5.6 Further reading 137

Part II General relativity and conformal geometry **139**

6 Conformal extensions of exact solutions 141
6.1 Preliminaries 141
6.2 The Minkowski spacetime 145
6.3 The de Sitter spacetime 155
6.4 The anti-de Sitter spacetime 159
6.5 Conformal extensions of static and stationary black hole spacetimes 163
6.6 Further reading 177

7 Asymptotic simplicity 178
7.1 Basic definitions 178
7.2 Other related definitions 181
7.3 Penrose's proposal 182
7.4 Further reading 183

8 The conformal Einstein field equations 184
8.1 A singular equation for the conformal metric 184
8.2 The metric regular conformal field equations 185
8.3 Frame and spinorial formulation of the conformal field equations 194
8.4 The extended conformal Einstein field equations 200
8.5 Further reading 209

9 Matter models 211
9.1 General properties of the conformal treatment of matter models 211
9.2 The Maxwell field 213
9.3 The scalar field 216
9.4 Perfect fluids 219
9.5 Further reading 221

10 Asymptotics 222
10.1 Basic set up: general structure of the conformal boundary 223
10.2 Peeling properties 226
10.3 The Newman-Penrose gauge 231
10.4 Other aspects of asymptotics 238
10.5 Further reading 241
Appendix: spin-weighted functions 241

Cambridge University Press
978-1-107-03389-4 - Conformal Methods in General Relativity
Juan A. Valiente Kroon
Frontmatter
More information

Part III Methods of the theory of partial differential equations **245**

11 The conformal constraint equations 247
11.1 General setting and basic formulae 247
11.2 Basic notions of elliptic equations 252
11.3 The Hamiltonian and momentum constraints 253
11.4 The conformal constraint equations 259
11.5 The constraints on compact manifolds 268
11.6 Asymptotically Euclidean manifolds 270
11.7 Hyperboloidal manifolds 284
11.8 Other methods for solving the constraint equations 286
11.9 Further reading 291
Appendix: some results of analysis 291

12 Methods of the theory of hyperbolic differential equations 294
12.1 Basic notions 294
12.2 Uniqueness and domains of dependence 301
12.3 Local existence results for symmetric hyperbolic systems 306
12.4 Local existence for boundary value problems 313
12.5 Local existence for characteristic initial value problems 319
12.6 Concluding remarks 328
12.7 Further reading 329
Appendix 329

13 Hyperbolic reductions 331
13.1 A model problem: the Maxwell equations on a fixed background 332
13.2 Hyperbolic reductions using gauge source functions 336
13.3 The subsidiary equations for the standard conformal field equations 354
13.4 Hyperbolic reductions using conformal Gaussian systems 366
13.5 Other hyperbolic reduction strategies 383
13.6 Further reading 385
Appendix A.1: the reduced Einstein field equations 386
Appendix A.2: differential forms 389

14 Causality and the Cauchy problem in general relativity 390
14.1 Basic elements of Lorentzian causality 390
14.2 PDE causality versus Lorentzian causality 395
14.3 Cauchy developments and maximal Cauchy developments 398
14.4 Stability of solutions 401
14.5 Causality and conformal geometry 402
14.6 Further reading 404

Part IV Applications **405**

15 De Sitter-like spacetimes 407
15.1 The de Sitter spacetime as a solution to the conformal field equations 408
15.2 Perturbations of initial data for the de Sitter spacetime 416
15.3 Global existence and stability using gauge source functions 420
15.4 Global existence and stability using conformal Gaussian systems 426
15.5 Extensions 434
15.6 Further reading 436

16 Minkowski-like spacetimes 437
16.1 The Minkowski spacetime and the conformal field equations 438
16.2 Perturbations of hyperboloidal data for the Minkowski spacetime 442
16.3 A priori structure of the conformal boundary 444
16.4 The proof of the main existence and stability result 450
16.5 Extensions and further reading 452

17 Anti-de Sitter-like spacetimes 454
17.1 General properties of anti-de Sitter-like spacetimes 455
17.2 The formulation of an initial boundary value problem 460
17.3 Covariant formulation of the boundary conditions 468
17.4 Other approaches to the construction of anti-de Sitter-like spacetimes 475
17.5 Further reading 475

18 Characteristic problems for the conformal field equations 477
18.1 Geometric and gauge aspects of the standard characteristic initial value problem 478
18.2 The conformal evolution equations in the standard characteristic initial value problem 485
18.3 A local existence result for characteristic problems 491
18.4 The asymptotic characteristic problem on a cone 496
18.5 Further reading 502

19 Static solutions 504
19.1 The static field equations 504
19.2 Analyticity at infinity 512
19.3 A regularity condition 517
19.4 Multipole moments 518
19.5 Uniqueness of the conformal structure of static metrics 522
19.6 Characterisation of static initial data 524
19.7 Further reading 525
Appendix 1: Hölder conditions 526
Appendix 2: the Cauchy-Kowalewskaya theorem 526

20 Spatial infinity 527
20.1 Cauchy data for the conformal field equations near spatial infinity 527
20.2 Massless and purely radiative spacetimes 531
20.3 A regular initial value problem at spatial infinity 538
20.4 Spatial infinity and peeling 554
20.5 Existence of asymptotically simple spacetimes 555
20.6 Obstructions to the smoothness of null infinity 556
20.7 Further reading 557
Appendix: properties of functions on the complex null cone 558

21 Perspectives 560
21.1 Stability of cosmological models 560
21.2 Stability of black hole spacetimes 562
21.3 Conformal methods and numerics 564
21.4 Computer algebra 567
21.5 Concluding remarks 568

References 569
Index 587

Cambridge University Press
978-1-107-03389-4 - Conformal Methods in General Relativity
Juan A. Valiente Kroon
Frontmatter
More information

Preface

This book discusses an approach to the study of global properties of solutions to the equations of general relativity, the Einstein field equations, in which the notion of conformal transformation plays a central role. The use of conformal transformations in differential geometry dates back, at least, to the work of *Hermann Weyl* in the 1920s.[1] Their application to global questions in general relativity, as presented in this book, stems from the seminal work of *Roger Penrose* in the 1960s in which the close connection between the global causal structure of the solutions to the equations of general relativity and conformal geometry was established.[2] Penrose's key insights are that the close relation between the propagation of the gravitational field and the structure of light cones which holds locally in a spacetime is also preserved in the case of large scales and that the asymptotic behaviour of the gravitational field can be conveniently analysed in terms of conformal extensions of the spacetime. In the following decade Penrose's ideas were polished, extended and absorbed into the mainstream research of general relativity by a considerable number of researchers[3] – finally leading to the influential notion of *asymptotic simplicity*. The subject reached its maturity when this *formal* theory was combined with the methods of the theory of partial differential equations (PDEs). This breakthrough is mainly due to the work of *Helmut Friedrich* in the early 1980s, who – through the *conformal Einstein field equations*[4] – showed that ideas of conformal geometry can be used to establish the existence of large classes of solutions to the Einstein field equations satisfying Penrose's notion of asymptotic simplicity. As a result of this work it is now clear that Penrose's original insights hold for large classes of spacetimes and not only for special explicitly known solutions.

This book develops the theory of the conformal Einstein field equations from the ground up and discusses their applications to the study of asymptotically simple spacetimes. Special attention is paid to results concerning the existence and stability of *de Sitter-like spacetimes*, the semiglobal existence and stability of *Minkowski-like spacetimes* using hyperboloidal Cauchy problems and the

[1] See Weyl (1968).
[2] See Penrose (1963, 1964).
[3] See e.g. Hawking and Ellis (1973); Geroch (1976).
[4] See Friedrich (1981a,b, 1983).

construction of *anti-de Sitter-like spacetimes* from initial boundary value problems. These results belong to the canon of modern mathematical relativity. In addition to their mathematical interest, they are of great physical relevance as they express, among other things, the internal consistency of general relativity and provide an approach for the global evaluation of spacetimes by means of numerical methods.

Why a book on the subject? The applications of conformal methods in general relativity constitute a mature subject with a number of *core results* which will withstand the pass of time. Still, it provides a number of challenging open questions whose resolution will strengthen its connections with other research strands in general relativity. This book aims at making the subject accessible to physicists and mathematicians alike who want to make use of conformal methods to analyse the global structure and properties of spacetimes. Hopefully, this book will provide an alternative to the use of original references while learning the subject or doing research.

Anyone who wants to engage with the subject of this book faces a number of challenges. To begin with, one has a vast literature spreading over more than 50 years. As it is to be expected from a living subject, the perspectives change through time, the importance of certain problems rise and wane and it is sometimes hard to differentiate the fundamental from the subsidiary. The combination of results from various references is often hindered by changing notation and conflicting conventions. Moreover, to appreciate and understand the results of the theory one requires a considerable amount of background material: conformal geometry, spinors, PDE theory, causal theory, etc. These methods are an essential part of the toolkit of a modern mathematical relativist. This book endeavours to bring together in a single volume all the required background material in a concise and coherent manner.

As a cautionary note, it should be mentioned what this book is not intended to be. This book is not an introductory book to general relativity. A certain familiarity with the subject is assumed from the outset – ideally at the level of Part I of R. Wald's book *General Relativity*.[5] This is also not a book on the applications of the theory of PDEs in general relativity. For this, there are other books available.[6] Also, although the Cauchy problem in general relativity is a leading theme, this book should not be viewed as a monograph on the topic – for this, I refer the interested reader to H. Ringström's monograph.[7]

I have endeavoured to write a book which not only serves as an *introduction to the subject* but also is a *tool for research*. With this idea in mind, I have striven to provide as much detail as possible of the arguments and calculations. However, at some stages supplying further details is neither possible nor desirable. Indeed, quoting the preface of J. L. Synge's classical book on general relativity: "There

[5] See Wald (1984).
[6] See e.g. Choquet-Bruhat (2008); Rendall (2008).
[7] See Ringström (2009).

are heavy calculations in the book, but there are places where the reader will find me sitting on the fence, whistling, instead of rushing into the fray"; see Synge (1960).[8] In an attempt to keep the readability and the length of the text under control, I have not endeavoured to provide completely general or optimal theorems – the attentive reader will realise this and is referred to the literature for further details, if required. As a picture is better than a thousand words, I have complemented the text with a considerable number of figures and diagrams which, I hope, will help to explain the content of the main text and provide useful visual models.

In writing this book, I have assumed the reader to have a certain mathematical maturity. Some basic knowledge of topology is needed – Appendix A in Wald's book contains the required background – as well as familiarity with basic tensorial calculus. I have, however, not assumed any prior knowledge of 2-spinors. The necessary toolkit is developed in the course of two chapters. Readers looking for a supplementary source on the topic are referred to J. Stewart's book.[9] The applications of conformal methods discussed in this book require certain knowledge of the theory of PDEs. I provide all the required material in a chapter of its own – nevertheless, some previous exposure to the basic ideas of the theory of PDEs is an advantage. Some arguments in the book make use of very concrete results of analysis. In these cases, I have included the necessary ideas in appendices to the various chapters.

[8] I am thankful to R. Beig for bringing my attention to this quote.
[9] See Stewart (1991).

Acknowledgements

First and foremost, I would like to acknowledge the influence that Helmut Friedrich had in the development of my understanding of the theory of the conformal Einstein field equations. He has been an inexhaustible source of knowledge and insights. I also would like to acknowledge the role that Robert Beig, Sergio Dain and Malcolm MacCallum have played in shaping complementary perspectives on the subject. I am also grateful for the extensive discussions with my collaborators Christian Lübbe and Jose Luis Jaramillo which have allowed to form, define and refine some of the points of view and concerns presented in this book.

I want to thank my students (Diego Carranza, Michael Cole, Edgar Gasperín, Adem Hursit and Jarrod Williams), collaborators and colleagues (Artur Alho, Robert Beig, Alberto Carrasco, Daniela Pugliese, Alfonso García-Parrado, David Hilditch, Jose Luis Jaramillo, Malcolm MacCallum, Belgin Seymenoglu and Claire Sommé) who have carefully read various parts of this book, thus providing valuable feedback and a seemingly never-ending list of typos. Of course, I am fully responsible for any mistakes or omissions left in the text. Finally, I would also like to thank Jean-Phillipe Nicolas for useful conversations on some of the topics covered in this book.

Although all the calculations presented in this book have been carried out by hand, the system `xAct` for `Mathematica` has been a valuable resource in case of trouble. I am thankful to Thomas Bäckdahl and Alfonso García-Parrado for their help at various stages with this system. I also thank Peter Hübner and Anil Zenginoglu for allowing me to use figures and diagrams from their publications.

I would like to thank my home department, the School of Mathematical Sciences of Queen Mary University of London, for the support granted during the course of the project and for the sabbatical term in spring 2014 when a considerable portion of this text took form. The primal notes which eventually turned into this book stem from a minicourse I held at the Centre of Mathematics of the University of Minho in Portugal in April 2011; further notes come from a minicourse held at the Institute of Mathematics and System Sciences of the Chinese Academy of Sciences in June 2013. I thank these institutions and particularly my hosts (Filipe Mena and Xiaoning Wu, respectively) for the invitation and hospitality. I also thank the Gravitational Physics Group of the

Faculty of Physics of the University of Vienna for their hospitality at various times during the last years.

Finally, I would like to express my deepest gratitude to my wife, Christiane Maria Losert-Valiente Kroon, for her constant support during the course of this project and for a careful and critical reading of the whole manuscript and for spotting an incredible amount of typos and mismatching indices in the equations. The conclusion of this book would have taken much longer without her help! Finally I would like to thank my father, Antonio Valiente, for instilling in me the desire of writing a book and my in-laws, Linde and Peter Losert, for the interest they have taken in this project.

Juan A. Valiente Kroon

Cambridge University Press
978-1-107-03389-4 - Conformal Methods in General Relativity
Juan A. Valiente Kroon
Frontmatter
More information

Symbols

$d^{*}_{\boldsymbol{ij}}$ alternative description of the components of the magnetic part of the rescaled Weyl tensor, page 264

$(\boldsymbol{e_a}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \Xi, s, L_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, T_{\boldsymbol{ab}})$ unknowns in the frame version of the standard conformal field equations, page 196

$(\boldsymbol{e_a}, \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \hat{L}_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, T_{\boldsymbol{ab}}, \Xi, d_{\boldsymbol{a}})$ unknowns in the frame version of the extended conformal field equations, page 205

$(\boldsymbol{e_{AA'}}, \Gamma_{\boldsymbol{AA'BC}}, \Xi, s, L_{\boldsymbol{AA'BB'}}, \phi_{\boldsymbol{ABCD}}, T_{\boldsymbol{AA'BB'}})$ unknowns in the spinorial version of the standard conformal field equations, page 199

$(\boldsymbol{e_{AA'}}, \hat{\Gamma}_{\boldsymbol{AA'BC}}, \hat{L}_{\boldsymbol{AA'BB'}}, \phi_{\boldsymbol{ABCD}}, T_{\boldsymbol{AA'BB'}}, \Xi, d_{\boldsymbol{AA'}})$ unknowns in the spinorial version of the extended conformal field equations, page 208

$(\boldsymbol{o}, \boldsymbol{\iota})$ spin basis in index-free notation, page 66

$(\hat{\Sigma}_{\boldsymbol{AA'BB'}}, \hat{\Xi}^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}, \hat{\Delta}_{\boldsymbol{CC'DD'BB'}}, \hat{\Lambda}_{\boldsymbol{BB'CD}}, \delta_{\boldsymbol{AA'}}, \varsigma_{\boldsymbol{AA'BB'}}, \gamma_{\boldsymbol{AA'BB'}})$ zero quantities in the spinorial extended conformal field equations, page 208

$(\hat{\Sigma}_{\boldsymbol{ab}}, \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \hat{\Delta}_{\boldsymbol{cdb}}, \Lambda_{\boldsymbol{bcd}}, \delta_{\boldsymbol{a}}, \gamma_{\boldsymbol{ab}}, \varsigma_{\boldsymbol{ab}})$ zero quantities in the frame extended conformal field equations, page 205

$(\mathcal{M}, \boldsymbol{g})$ generic spacetime, page 45

$(\mathcal{U}, \varphi)$ coordinate chart, page 28

$(\Sigma_{\boldsymbol{AA'BB'}}, \Xi^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}, Z_{\boldsymbol{AA'BB'}}, Z_{\boldsymbol{AA'}}, \Delta_{\boldsymbol{CDBB'}}, \Lambda_{\boldsymbol{BB'CD}}, Z, M_{\boldsymbol{AA'}})$ zero quantities in the spinorial version of the standard conformal equations, page 199

$(\Sigma_{\boldsymbol{ab}}, \Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, Z_{\boldsymbol{ab}}, Z_{\boldsymbol{a}}, \Delta_{\boldsymbol{cdb}}, \Lambda_{\boldsymbol{bcd}}, Z, M_{\boldsymbol{a}})$ zero quantities in the frame version of the standard conformal field equations, page 196

$(g_{ab}, \Xi, s, L_{ab}, d^{a}{}_{bcd}, T_{ab})$ unknowns in the metric standard conformal field equations, page 191

$(h_{ij},\ s_{ij},\ \zeta,\ \varsigma)$ unknowns in the conformal static equations, page 511

(o^A, ι^A) spin basis in abstract index notation, page 71

$(u, r, \theta^{\mathcal{A}})$ Bondi coordinates, page 236

$(x(\tau), \boldsymbol{\beta}(\tau))$ conformal geodesic with parameter τ, page 127

(x^{μ}) local coordinates in a four-dimensional manifold, page 28

$[\nabla_a, \nabla_b]$ commutator of covariant derivatives, page 39

$[[\boldsymbol{\xi}, \boldsymbol{\eta}]]$ antisymmetric product of $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathfrak{S}$, page 65

$[\boldsymbol{g}]$ conformal class of the metric $\boldsymbol{g}$, page 113

$[\boldsymbol{u}, \boldsymbol{v}]$ commutator of the vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$, page 34

$\alpha_{\boldsymbol{a}}$, $\beta_{\boldsymbol{a}}$, $\omega_{\boldsymbol{a}}, \ldots$ components of the covectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\omega}, \ldots$ with respect to the frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$, page 51

α_a, β_a, $\omega_a, \ldots$ generic covectors in abstract index notation

$\approx$ diffeomorphism between sets, page 27

$\bar{\xi}^{A'}$, $\bar{\eta}_{A'}, \ldots$ complex conjugates of the spinors ξ^A, $\eta_B, \ldots$, page 72

β^2 norm of the covector $\tilde{\boldsymbol{\beta}}$, page 134

β_a covector associated to a conformal geodesic in abstract index notation, page 127

$\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\omega}$, ... generic covectors in index-free notation

$\boldsymbol{\beta}$ covector associated to a conformal geodesic in index-free notation, page 127

$\boldsymbol{\chi}$ Weingarten map, page 56

$\boldsymbol{D}$ generic three-dimensional connection in index-free notation

$\boldsymbol{d}$ rescaled conformal geodesic covector, page 201

$\boldsymbol{\delta}$ Euclidean metric on $\mathbb{R}^3$, page 143

$\boldsymbol{e}$, $\boldsymbol{e}_{\boldsymbol{AB}}$ space spinor irreducible components of the frame vector $\boldsymbol{e}_{\boldsymbol{AA'}}$, page 104

$\boldsymbol{\ell}$ three-dimensional Lorentzian metric on the conformal boundary of an anti-de Sitter-like spacetime, page 456

$\boldsymbol{f}$ covector defining a Weyl connection in index-free notation, page 119

$\boldsymbol{f}$ unphysical conformal geodesic covector, page 201

$\boldsymbol{g}$ generic four-dimensional Lorentzian metric tensor in index-free notation

$\boldsymbol{g}^\sharp$ generic contravariant four-dimensional Lorentzian metric tensor in index-free notation

$\boldsymbol{g}_{\mathscr{E}}$ standard metric on the Einstein cylinder, page 144

$\boldsymbol{\gamma}$ metric in the quotient manifold, page 141

$\boldsymbol{h}$ generic (negative definite) Riemannian three-dimensional metric

$\boldsymbol{\hbar}$ standard metric on the unit 3-sphere, page 142

$\boldsymbol{K}$ extrinsic curvature tensor of a hypersurface in index-free notation, page 61

$\boldsymbol{k}$ intrinsic metric of compact two-dimensional surfaces

$\boldsymbol{M}$, $\boldsymbol{N}, \ldots$ generic higher rank tensors in index-free notation

$\boldsymbol{N}$ tangent to the generators of null infinity

$\boldsymbol{\nabla}$, $\bar{\boldsymbol{\nabla}}$ generic linear connections in index-free notation, page 38

$\boldsymbol{\nu}$ unit normal to a hypersurface $\mathcal{S}$, page 54

$\boldsymbol{\omega}$, $\boldsymbol{\omega}^{\boldsymbol{AB}}$ space spinor irreducible components of the frame covector $\boldsymbol{\omega}^{\boldsymbol{AA'}}$, page 104

$\boldsymbol{\partial}_\mu$ coordinate basis vector

$\boldsymbol{Q}$ transition tensor between connections in index-free notation, page 42

$\boldsymbol{q}$ intrinsic metric of null infinity

$\boldsymbol{\Sigma}$ torsion tensor of a connection $\boldsymbol{\nabla}$ in index-free notation, page 39

$\boldsymbol{\sigma}$ standard metric on the unit 2-sphere

$\boldsymbol{\sigma}_{\mathbf{L}}(\boldsymbol{\xi})$ symbol of a differential operator $\mathbf{L}$, page 252

$\boldsymbol{t}$ vector field generating a timelike congruence

$\boldsymbol{\tau}$ vector counterpart of the spinor $\tau_{AA'}$, page 102

$\boldsymbol{v}$, $\boldsymbol{u}$, $\boldsymbol{w}, \ldots$ generic vectors in index-free notation

ς shear tensor, page 226
$\boldsymbol{z}$, $\boldsymbol{\zeta}$ deviation vector and covector, respectively, page 135
$\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}$ conformal Killing operator of the metric $\boldsymbol{h}$, page 257
$\breve{\mathbf{u}}$ perturbation quantity in an evolution system
$\chi_{(AB)CD}$ spinorial counterpart of the Weingarten tensor
χ_{AB} spinorial counterpart of the acceleration vector
$\circ$ composition of functions, page 36
$\coprod$ disjoint union of sets
$\delta(i)$ Dirac's delta, page 279
$\Delta_{\boldsymbol{h}}$ Laplacian operator of the Riemannian metric $\boldsymbol{h}$
$\delta_\mu{}^\nu$, $\delta_a{}^b$, $\delta_i{}^j$, $\delta_\alpha{}^\beta$, $\delta_A{}^B$, $\delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}$, $\delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$, $\delta_{\boldsymbol{i}}{}^{\boldsymbol{j}}$ Kronecker's delta
$\delta_{\alpha\beta}$ components of the three-dimensional Euclidean metric in Cartesian coordinates, page 47
$\delta_{\boldsymbol{AB}}$ Sen connection on a timelike conformal boundary, page 471
$\delta_{\boldsymbol{ij}}$ components of a three-dimensional Riemannian metric with respect to an orthonormal basis, page 45
$\dot{\boldsymbol{\gamma}}(\mathrm{s})$ tangent vector to a curve, page 30
$\dot{\boldsymbol{x}}(\mathrm{s})$ alternative notation for the tangent vector to a curve, page 30
$\epsilon = \pm 1$ encodes the causal character of a hypersurface, page 54
$\epsilon_{\boldsymbol{abcd}}$ components of the volume form with respect to an orthonormal basis
$\epsilon_{\boldsymbol{AB}}$, $\epsilon^{\boldsymbol{AB}}$ components of the spinors ϵ_{AB}, ϵ^{AB} with respect to a spin basis, page 71
$\epsilon_{A'B'}$, $\epsilon^{A'B'}$ complex conjugates of the spinors ϵ_{AB}, ϵ^{AB}
$\epsilon_{AA'BB'CC'DD'}$ spinorial counterpart of the volume form, page 78
ϵ_{ABCDEF} spinorial counterpart of the three-dimensional volume form, page 105
ϵ_{abcd} volume form of a metric g_{ab}, page 49
ϵ_{AB}, ϵ^{AB} antisymmetric spinors, page 67
$\equiv$ definition
$\eta_{\boldsymbol{ABCD}}$ components of the electric part of the Weyl spinor, page 373
$\eta_{\boldsymbol{ab}}$ components of a four-dimensional Lorentzian metric with respect to an orthonormal basis, page 45
$\eta_{\mu\nu}$ components of the Minkowski metric tensor in Cartesian coordinates, page 47
ð, $\bar{\text{ð}}$ eth and eth-bar operators, page 241
exp exponential map, page 275
Γ geodesic distance, page 276
$\gamma(\mathrm{s})$ curve in a manifold with parameter s, page 30
$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ connection coefficients of $\boldsymbol{\nabla}$ with respect to $\{\boldsymbol{e_a}\}$
$\gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}$ connection coefficients of the three-dimensional connection $\boldsymbol{D}$ with respect to the frame $\{\boldsymbol{e_i}\}$, page 59
$\Gamma_\mu{}^\nu{}_\lambda$ Christoffel symbols of the metric $\boldsymbol{g}$ in the coordinates (x^μ)
$\Gamma_{\boldsymbol{A'A'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}}$ spinorial counterpart of the connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$, page 82
$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$ reduced spin connection coefficients, page 82

$\Gamma_{\boldsymbol{ABCD}}$ space spinor counterpart of the reduced spin connection coefficients $\Gamma_{\boldsymbol{AA'CD}}$, page 107
$\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}$ reduced spatial spin connection coefficients, page 109
$\gamma_{\boldsymbol{AB}}{}^{\boldsymbol{CD}}{}_{\boldsymbol{EF}}$ spinorial counterpart of the three-dimensional connection coefficients $\gamma_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{k}}$, page 109
$\grave{\Omega}$, $\check{\Omega}$ massless and, respectively, massive part of the conformal factor associated to Euclidean initial data sets, page 529
$\hat{\boldsymbol{\nabla}}$ generic Weyl connection in index-free notation, page 119
$\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ connection coefficients of a Weyl connection $\hat{\boldsymbol{\nabla}}$, page 119
$\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$ reduced Weyl connection spin coefficients, page 206
$\hat{\nabla}_a$ generic Weyl connection in abstract index notation, page 119
$\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ Weyl connection algebraic curvature, page 203
$\hat{\rho}_{\boldsymbol{ABCC'DD'}}$ Weyl connection reduced spinorial algebraic curvature, page 207
$\hat{P}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ Weyl connection geometric curvature, page 203
$\hat{P}_{\boldsymbol{ABCC'DD'}}$ Weyl connection reduced spinorial geometric curvature, page 207
κ conformal factor associated to the construction of the cylinder at spatial infinity, page 541
Λ Newman-Penrose Ricci scalar, page 87
λ cosmological constant, page 2
$\Lambda_{(\boldsymbol{ABCD})}$, $\Lambda_{\boldsymbol{AB}}$ irreducible components of the spinorial Bianchi equation, page 351
$\langle \boldsymbol{\omega}, \boldsymbol{v} \rangle$ action of the covector $\boldsymbol{\omega}$ on the vector $\boldsymbol{v}$
$\langle \boldsymbol{t} \rangle^{\perp}|_p$ subspace orthogonal to $\boldsymbol{t}$, page 55
$\langle \boldsymbol{t} \rangle$ one-dimensional subspace spanned by $\boldsymbol{t}$, page 55
$\langle\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle\rangle$ Hermitian product of $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{S}$, page 94
$[\![\nabla_a, \nabla_b]\!]$ modified commutator of covariant derivatives, page 40
$\mathbb{H}^n$ n-dimensional half Euclidean space, page 29
$\mathbb{R}^+$ non-negative real numbers
$\mathbb{R}^2$ Euclidean plane
$\mathbb{R}^n$ n-dimensional Euclidean space
$\mathbb{S}^2$ 2-sphere
$\mathbb{S}^3$ three-dimensional unit sphere, page 142
$\mathbf{A}^*$ transpose of the complex conjugate of the matrix $\mathbf{A}$
$\mathbf{A}^3$ normal matrix in an initial boundary value problem, page 314
$\mathbf{A}^\mu$ symmetric matrices in a symmetric hyperbolic system, page 294
$\mathbf{d}$ exterior derivative (differential), page 31
$\mathbf{d}x^\mu$ coordinate basis covector
$\mathbf{L}$ generic differential operator
$\mathbf{L}^*$ formal adjoint of the differential operator $\mathbf{L}$
$\mathbf{L}_{\boldsymbol{h}}$ Yamabe operator, page 256
$\mathbf{T}$ map associated to the prescription of boundary conditions in an initial boundary value problem, page 314
$\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}, \ldots$ $\mathbb{C}^N$-valued functions

$\mathcal{B}_a(p)$ ball of radius $a > 0$ centred at the point p
$\mathcal{C}_p$ null cone at a point $p \in \mathcal{M}$, page 45
$\mathcal{C}_p^+$, $\mathcal{C}_p^-$ future and, respectively, past null cone at a point $p \in \mathcal{M}$, page 45
$\mathcal{D}$ a generic derivation, page 30
$\mathcal{D}_{AB}$ Sen connection of $\nabla_{AA'}$ induced by $\tau_{AA'}$, page 105
$\mathcal{E}$ corner in an initial boundary value problem, page 314
$\mathcal{G}$ generic lens-shaped domain, page 301
$\mathcal{H}_k$ standard hyperboloids, page 154
$\mathcal{I}$ cylinder at spatial infinity, page 542
$\mathcal{I}^0$ intersection of the cylinder at spatial infinity with a Cauchy initial hypersurface, page 542
$\mathcal{I}^\pm$ critical sets where null infinity touches spatial infinity, page 542
$\mathcal{M}$, $\mathcal{N}$ generic (unphysical) spacetime manifolds
$\mathcal{N}$, $\mathcal{N}'$ initial null hypersurfaces in a characteristic problem, page 320
$\mathcal{N}_i$ complex null cone at i, page 522
$\mathcal{N}_{\mathbb{C}}(i)$ complexification of the null cone through i, page 532
$\mathcal{P}$ covariant derivative in the direction of $\tau_{AA'}$, page 105
$\mathcal{Q}$ generic quotient manifold, page 141
$\mathcal{R}$ generic subset of a hypersurface $\mathcal{S}$
$\mathcal{S}$ generic hypersurface on a manifold $\mathcal{M}$
$\mathcal{T}$ timelike boundary, page 314
$\mathcal{U}$, $\mathcal{V}$ generic open subsets of a manifold or $\mathbb{R}^n$
$\mathcal{U}_{\mathbb{C}}$ complexification of a neighbourhood $\mathcal{U}$ of the point at infinity, page 532
$\mathcal{Z}$ intersection of initial null hypersurfaces in a characteristic problem, page 320
$\mathfrak{S}$ complex vector space, page 65
$\mathfrak{S}(\mathcal{M})$ spin structure (spin bundle) over $\mathcal{M}$, page 81
$\mathfrak{S}(\mathcal{S})$ space spinor structure over a three-dimensional manifold $\mathcal{S}$, page 101
$\mathfrak{S}^*$ dual of the complex vector space $\mathfrak{S}$, page 65
$\mathfrak{S}^\bullet(\mathcal{M})$, $\mathfrak{S}_A(\mathcal{M})$, $\mathfrak{S}^A(\mathcal{M})$, $\mathfrak{S}_{AA'}{}^B(\mathcal{M}), \ldots$ various spin bundles over $\mathcal{M}$
$\mathfrak{S}^\bullet$ spin algebra, page 66
$\mathfrak{S}^A$, $\mathfrak{S}_A$, ... alternative notation for the vector spaces $\mathfrak{S}$, $\mathfrak{S}^*, \ldots$, page 66
$\mathfrak{S}^{A'}$, $\mathfrak{S}_{A'B'}$, ... complex conjugates of the spaces $\mathfrak{S}^A$, $\mathfrak{S}_{AB}, \ldots$, page 72
$\mathfrak{T}^\bullet(\mathcal{M})$ tensor bundle over $\mathcal{M}$, page 34
$\mathfrak{T}^a(\mathcal{M})$ alternative notation for the tangent bundle over $\mathcal{M}$, page 36
$\mathfrak{T}^{a_1\cdots a_k}{}_{b_1\cdots b_l}(\mathcal{M})$ alternative notation for the tensor bundle over $\mathcal{M}$, page 36
$\mathfrak{T}_a(\mathcal{M})$ alternative notation for the cotangent bundle over $\mathcal{M}$, page 36
$\mathfrak{X}(\mathcal{M})$ set of of scalar fields over $\mathcal{M}$, page 30
$\mathring{\mathbf{u}}$ background quantity in an evolution system
$\mathscr{C}$ generic cut of null infinity
$\mathscr{C}_\star$ fiduciary cut of null infinity
$\mathscr{E}$ extension operator of functions between Sobolev spaces, page 308
$\mathscr{I}$ part of the conformal boundary that is a hypersurface, page 178
$\mathscr{I}^\pm$ future and, respectively, past null infinity

$\mathscr{N}_i^+$, $\mathscr{N}_i^-$ null cones generated by the null geodesics through i, page 531
$\mathscr{N}_u$ outgoing null hypersurface associated to the retarded time u
$\mathscr{R}_{\boldsymbol{h}}$ linearised Ricci operator, page 289
$\mathscr{Z}$ generic intersection of null infinity with a null hypersurface
$\operatorname{int}\mathcal{A}$ topological interior of the set $\mathcal{A}$, page 397
i square root of -1
$\mu_{\boldsymbol{ABCD}}$ components of the magnetic part of the Weyl spinor, page 373
$\nabla_{\boldsymbol{a}}$ covariant directional derivative in the direction of $\boldsymbol{e_a}$, page 51
$\nabla_{\boldsymbol{u}}\boldsymbol{v}$ covariant derivative of $\boldsymbol{v}$ with respect to $\boldsymbol{u}$, page 38
∇_a, $\bar{\nabla}_a$ generic linear connections in abstract index notation, page 38
$\nabla_{\boldsymbol{AA'}}$ directional spinorial covariant derivative, page 82
$\nabla_{AA'}$, $\tilde{\nabla}_{AA'}, \ldots$ spinor covariant derivatives, page 81
∇_{AB} space spinor counterpart of $\nabla_{AA'}$, page 105
Ω generic three-dimensional conformal factor
$\oplus$ direct sum
$\otimes$ tensor product between tensors or tensor spaces
$\overline{\mathcal{A}}$ topological closure of the set $\mathcal{A}$, page 394
$\| \mathbf{u} \|_{\mathcal{S},m}$ Sobolev norm of order m of a function over $\mathcal{S}$, page 306
$\partial\mathbb{H}^n$ boundary of the n-dimensional half Euclidean space, page 29
$\partial\mathcal{M}$ boundary of $\mathcal{M}$
ϕ unphysical conformally coupled scalar field, page 216
ϕ_0 radiation field in the asymptotic characteristic problem on a cone, page 500
$\Phi_{ABA'B'}$ spinorial counterpart of the trace-free Ricci tensor, page 89
Φ_{ab} trace-free Ricci tensor of a connection ∇_a in abstract index notation, page 48
ϕ_{AB} unphysical Maxwell spinor, page 215
Π generic distribution, page 55
$\Pi\,|_p$ hyperplane induced by a distribution at a point $p \in \mathcal{M}$, page 55
$\pounds_{\boldsymbol{v}}$ Lie derivative in the direction of $\boldsymbol{v}$, page 37
Ψ_{ABCD} Weyl spinor, page 87
ρ boundary-defining function, page 285
ρ polar radial coordinate, page 514
ρ^α three-dimensional unit position vector, page 514
$\rho^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}$ reduced spinorial algebraic curvature, page 198
$\rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ components of the algebraic curvature, page 195
$\rho^{AA'}$ spatial spinor used to introduce a $1+1+2$ spinor formalism, page 464
$\boldsymbol{Ric}$, $\boldsymbol{Ric}[\boldsymbol{g}]$ Ricci tensor of a connection $\boldsymbol{\nabla}$ in index-free notation, page 48
$\boldsymbol{Riem}$ Riemann curvature tensor of a connection $\boldsymbol{\nabla}$ in index-free notation, page 40
$\boldsymbol{Schouten}$, $\boldsymbol{Schouten}[\boldsymbol{g}]$ Schouten tensor of a connection $\boldsymbol{\nabla}$ in index-free notation, page 48
σ Newman-Penrose spin connection coefficient corresponding to $\Gamma_{\mathbf{01'00}}$
$\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}$, $\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}$ spacetime Infeld-van der Waerden symbols, page 74

Cambridge University Press
978-1-107-03389-4 - Conformal Methods in General Relativity
Juan A. Valiente Kroon
Frontmatter
More information

$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ components of the torsion tensor with respect to an orthonormal frame, page 53
$\sigma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$, $\Pi^{\boldsymbol{k}}{}_{\boldsymbol{lij}}$, $\pi_{\boldsymbol{klij}}$ components of the three-dimensional torsion, geometric and algebraic curvatures, page 264
$\sigma_{\boldsymbol{i}}{}^{\boldsymbol{AB}}$, $\sigma^{\boldsymbol{i}}{}_{\boldsymbol{AB}}$ spatial Infeld-van der Waerden symbols, page 99
$\Sigma_a{}^c{}_b$ torsion tensor of a connection ∇_a in abstract index notation, page 39
$\simeq$ equality at the conformal boundary
$\square$ D'Alembertian operator, page 89
$\square_{AB}$ box commutator, page 89
$\overset{\star}{\simeq}$ equality at a fiduciary cut of null infinity
$\tau_{AA'}$ privileged timelike spinor inducing a space spinor formalism, page 102
Θ conformal factor associated to a conformal geodesic, page 132
$\theta = (\theta^{\mathcal{A}})$ local coordinates on $\mathbb{S}^2$
$\Theta_{\boldsymbol{ABCD}}$ space spinor counterpart of the components of the Schouten tensor of a Weyl connection, page 373
$\tilde{\boldsymbol{\eta}}$ Minkowski metric
$\tilde{\boldsymbol{g}}_{\mathscr{E}}$ metric of the anti-de Sitter spacetime, page 159
$\tilde{\boldsymbol{g}}_{\mathscr{S}}$ metric of the Schwarzschild spacetime, page 163
$\tilde{\boldsymbol{g}}_{dS}$ metric of the de Sitter spacetime, page 155
$\tilde{\mathcal{E}}_k$ asymptotic ends of asymptotically Euclidean manifold $\tilde{\mathcal{S}}$, page 272
$\tilde{\mathcal{F}}_{ab}$ self-dual Faraday tensor, page 213
$\tilde{\mathcal{M}}$ generic (physical) spacetime manifold
$\tilde{\phi}$ physical conformally coupled scalar field, page 216
$\tilde{\phi}_{AB}$ physical Maxwell spinor, page 215
$\tilde{\varrho}$ density of a perfect fluid, page 219
$\tilde{\varrho}$ energy density, page 254
$\tilde{F}_{ab}$ physical Faraday tensor, page 213
$\tilde{j}_k$ energy flux vector, page 254
$\tilde{p}$ pressure of a perfect fluid, page 219
$\tilde{T}_{ab}$ physical energy-momentum tensor
$\tilde{u}^a$ physical 4-velocity of a perfect fluid, page 219
$\underline{x}$ spatial coordinates (x^1, x^2, x^3)
Υ_a logarithmic gradient of a conformal factor, page 116
$\Upsilon_{AA'}$ spinorial counterpart of the logarithmic gradient of a conformal factor, page 123
φ^* pull-back, page 36
φ_* push-forward, page 36
$\varpi_{\boldsymbol{AA'}}$ components of $\varpi_{AA'}$ with respect to a spin basis, page 95
$\varpi_{AA'}$ Hermitian spinor assocated to a Hermitian inner product, page 95
ϱ conformally rescaled density of a perfect fluid, page 220
ϱ unphysical energy density, page 255
$\boldsymbol{Weyl}$, $\boldsymbol{Weyl}[\boldsymbol{g}]$ Weyl tensor of a connection $\boldsymbol{\nabla}$ in index-free notation, page 48
$\xi^{\boldsymbol{A}}$, $\eta_{\boldsymbol{A}}, \ldots$ components of the spinors ξ^A, $\eta_A, \ldots$ with respect to a spin basis

ξ^A, $\eta_A, \ldots$ generic spinors in abstract-index notation
$\xi_{\boldsymbol{ABCC}}$, $\chi_{\boldsymbol{ABCD}}$ real and imaginary parts of $\Gamma_{\boldsymbol{ABCD}}$, page 107
Ξ_{ij}, $S_{\boldsymbol{i}}$, S_{ij}, H_{kij} zero quantities associated to the conformal static field equations, page 511
$\zeta_0, \ldots \zeta_4$ components of the spin-2 zero-rest mass field ζ_{ABCD}, page 551
ζ_{ABCD} spin-2 zero-rest mass field, page 551
$\{\boldsymbol{c_i}\}$ global orthonormal frame on $\mathbb{S}^3$, page 142
$\{\boldsymbol{e_a}\}$ vector basis in index-free notation, page 31
$\{\boldsymbol{\omega^a}\}$ covector basis in index-free notation, page 31
$\{\mathcal{S}_t\}_{t\in\mathbb{R}}$ foliation of $\mathcal{M}$, page 54
$\{\boldsymbol{e_i}\}$ three-dimensional vector basis in index-free notation, page 59
$\{\boldsymbol{e_{AA'}}\}$ alternative index-free notation for the Newman-Penrose null tetrad, page 79
$\{\boldsymbol{e_{AB}}\}$, $\{\boldsymbol{\omega^{AB}}\}$ three-dimensional basis and cobasis with spin frame indices, page 109
$\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ Newman-Penrose null tetrad in index-free notation, page 77
$\{\boldsymbol{\omega^i}\}$ three-dimensional covector basis in index-free notation, page 59
$\{\boldsymbol{\omega^{AA'}}\}$ soldering form, page 79
$\{\epsilon_{\boldsymbol{A}}{}^A\}$, $\{\epsilon^{\boldsymbol{A}}{}_A\}$ alternative abstract index notation for a spin basis and its dual, page 71
$\{\omega^{\boldsymbol{a}}{}_a\}$ covector basis in abstract index notation, page 36
$\{\omega^{\boldsymbol{i}}{}_i\}$ three-dimensional covector basis in index-free notation, page 59
$\{e_{\boldsymbol{a}}{}^a\}$ vector basis in abstract index notation, page 36
$\{e_{\boldsymbol{i}}{}^i\}$ three-dimensional vector basis in abstract index notation, page 59
$\{l^a, n^a, m^a, \bar{m}^a\}$ Newman-Penrose null tetrad in abstract index notation, page 77
$\{m,\, m_\alpha,\, m_{\alpha_1\alpha_2}, \ldots\}$ sequence of multipole moments of a static spacetime, page 519
b_{ABCD} Cotton spinor, page 512
C^*_p characteristic set of a symmetric hyperbolic system at the point p, page 297
C^∞ class of infinitely differentiable (smooth) functions
$C^\infty(\mathbb{R}^3, \mathbb{C}^N)$ space of smooth functions from $\mathbb{R}^3$ to $\mathbb{C}^N$, page 306
$C^c{}_{dab}$ Weyl tensor of a connection ∇_a in abstract index notation, page 48
C^k class of k-times differentiable functions
$C^k(\mathbb{R}^3, \mathbb{C}^N)$ set of C^k functions from $\mathbb{R}^3$ to $\mathbb{C}^N$, page 307
$C^k\big([0,T]; H^m(\mathbb{R}^3, \mathbb{C}^N)\big)$ set of C^k functions from $[0,T]$ to $H^m(\mathbb{R}^3, \mathbb{C}^N)$, page 307
D bounded open subset of $H^m(\mathbb{R}^3, \mathbb{C}^N)$ such that for $\mathbf{w} \in D$ the matrix $\mathbf{A}^0(0, \underline{x}, \mathbf{w})$ is positive definite bounded away from zero by $\delta > 0$ for all $p \in \mathbb{R}^3$, page 309
$D(\mathcal{R})$ domain of dependence of $\mathcal{R}$, page 304
D, Δ, δ, $\bar{\delta}$ Newman-Penrose directional covariant derivatives, page 92
$D^\pm(\mathcal{A})$, $D(\mathcal{A})$ future/past and total domain of dependence of a set $\mathcal{A}$, page 392

$d^a{}_{bcd}$ rescaled Weyl tensor, page 188
$d_{\boldsymbol{a}}$ components of the rescaled physical conformal geodesics covector, page 203
$D_{\boldsymbol{i}}$ three-dimensional directional covariant derivative in the direction of $\boldsymbol{e}_{\boldsymbol{i}}$, page 59
D_i generic three-dimensional connection in abstract index notation
$D_{\boldsymbol{AB}}$ three-dimensional covariant directional derivative, page 109
$d_{\boldsymbol{ij}}$, $d_{\boldsymbol{ijk}}$ components of the electric and magnetic parts of the rescaled Weyl tensor, page 261
D_{AB} spinorial counterpart of a three-dimensional Levi-Civita connection $\boldsymbol{D}$, page 106
$F^{\boldsymbol{a}}(x)$, $F^\mu(x)$ coordinate gauge source functions, page 339
$f_{\boldsymbol{a}}$ components of the unphysical conformal geodesics covector, page 203
f_a covector defining a Weyl connection in abstract index notation, page 119
$F_{\boldsymbol{AB}}(x)$ frame gauge source functions, page 345
F_{ab} unphysical Faraday tensor, page 214
g^{ab} generic contravariant four-dimensional Lorentzian metric tensor in abstract index notation
G_{ab} Einstein tensor of a metric g_{ab}
g_{ab} generic four-dimensional Lorentzian metric tensor in abstract index notation
$H^\pm(\mathcal{A})$, $H(\mathcal{A})$ future/past and total Cauchy horizons of the set $\mathcal{A}$, page 394
$H^m(\mathbb{R}^3, \mathbb{C}^N)$ Sobolev space of order m of functions from $\mathbb{R}^3$ to $\mathbb{C}^N$, page 307
$h_a{}^b$ projector associated to a distribution Π, page 55
$h_{\boldsymbol{ABCD}}$ components of h_{ABCD} with respect to a spin frame $\{\epsilon_{\boldsymbol{A}}{}^A\}$, page 99
$h_{AA'}{}^{BB'}$ spinorial counterpart of the projector $h_{\boldsymbol{a}}{}^{\boldsymbol{b}}$, page 98
h_{ABCD} space spinor counterpart of $h_{AA'}{}^{BB'}$ and of a three-dimensional Riemannian metric, page 98
I generic interval in $\mathbb{R}$
i^0 spatial infinity
$i^\pm$ future and, respectively, past timelike infinity
$I^\pm(\mathcal{U})$ chronological future and, respectively, past of a set $\mathcal{U}$, page 391
$J^+(o, \mathcal{M}')$ set consisting of o and all points of $\mathcal{M}'$ which can be joined to o by a causal curve in $\mathcal{M}'$, page 497
$J^\pm(\mathcal{U})$ causal future and, respectively, past of a set $\mathcal{U}$, page 391
j_k unphysical flux vector, page 255
$J_{\boldsymbol{jk}}$, $J_{\boldsymbol{j}}$ normal components of the rescaled Cotton tensor, page 262
K_{ij} extrinsic curvature tensor of a hypersurface in abstract index notation, page 61
L_{ab} Schouten tensor of a connection ∇_a in abstract index notation, page 48
l_{ij} three-dimensional Schouten tensor, page 60
$p \prec\!\prec q$ timelike related points, page 391
p conformally rescaled pressure of a perfect fluid, page 220
$p \prec q$ strictly causally related points, page 391

Cambridge University Press
978-1-107-03389-4 - Conformal Methods in General Relativity
Juan A. Valiente Kroon
Frontmatter
More information

$p \preceq q$ causally related points, page 391
$P^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}$ reduced spinorial geometric curvature, page 198
$P^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ components of the geometric curvature, page 194
$P^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}}$ spinorial geometric curvature, page 197
$P_n^{(\alpha,\beta)}(\tau)$ Jacobi polynomial of degree n with parameters (α,β), page 553
$Q_a{}^b{}_c$ transition tensor between connections in abstract index notation, page 42
r three-dimensional Ricci scalar, page 60
$R(x)$ conformal gauge source function, page 348
R, $R[\boldsymbol{g}]$ Ricci scalar of a connection ∇_a, page 48
$R^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ components of the Riemann tensor with respect to an orthonormal frame, page 53
$R^d{}_{cab}$ Riemann curvature tensor of a connection ∇_a in abstract index notation, page 40
$r^k{}_{lij}$ three-dimensional Riemann curvature tensor in abstract index notation, page 60
$r_{ABCDEFGH}$ spinorial counterpart of the three-dimensional Riemann curvature tensor, page 110
R_{ab} Ricci tensor of a connection ∇_a in abstract index notation, page 48
r_{ACEFGH}, r_{ABCE} reduced three-dimensional curvature spinors, page 110
$R_{CC'DD'AA'BB'}$ spinorial counterpart of the Riemann curvature tensor, page 86
$R_{CDAA'BB'}$ reduced Riemann curvature spinor, page 86
r_{ij} three-dimensional Ricci tensor in abstract index notation, page 60
s the Friedrich scalar, page 186
s_{ABCD} spinorial counterpart of the three-dimensional trace-free Ricci tensor, page 110
s_{ij} three-dimensional trace-free Ricci tensor, page 60
$SO(3)$ three-dimensional special orthogonal group
$T(\mathcal{M})$ tangent bundle over $\mathcal{M}$, page 34
$T|_p(\mathcal{M})$ tangent space at a point $p \in \mathcal{M}$, page 31
$T^*(\mathcal{M})$ cotangent bundle over $\mathcal{M}$, page 34
$T^*|_p(\mathcal{M})$ cotangent space at a point $p \in \mathcal{M}$, page 31
$T^\bullet|_p(\mathcal{M})$ tensor algebra at $p \in \mathcal{M}$, page 33
$T^k_l|_p(\mathcal{M})$ space of (k,l)-tensors at the point $p \in \mathcal{M}$, page 33
$T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ arbitrary (k,l)-tensor in abstract index notation
T_{ab} unphysical energy-momentum tensor
T_{cdb} rescaled Cotton tensor, page 189
u retarded time
U, $X^{\mathcal{A}}$, ω, $\xi^{\mathcal{A}}$ components of an adapted frame in the asymptotic characteristic problem, page 482
u, v retarded and, respectively, advanced time coordinates
$u^{\boldsymbol{a}}$, $v^{\boldsymbol{a}}$, $w^{\boldsymbol{a}}, \ldots$ components of the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ with respect to the coframe $\{\boldsymbol{\omega}^{\boldsymbol{a}}\}$, page 51

u^a unphysical 4-velocity of a perfect fluid, page 220
u^a, v^a, $w^a, \ldots$ generic vectors in abstract index notation
v norm of a static Killing vector, page 504
$x(\mathrm{s})$ alternative notation for a curve with parameter s, page 30
X_{CDAB}, $Y_{CDA'B'}$ curvature spinors, page 86
$Y[\boldsymbol{h}]$ Yamabe invariant, page 280
Y_{abc} four-dimensional Cotton tensor, page 116
y_{ijk} three-dimensional Cotton tensor, page 118
y_{ij} three-dimensional Cotton-York tensor, page 118
$z_{\boldsymbol{AA'}}$, z, $z_{(AB)}$ spacetime and space spinor components of the spinorial counterpart of the deviation vector of a congruence of conformal geodesics, page 383
${}^*F_{ab}$ Hodge dual of an antisymmetric tensor F_{ab}, page 50
${}^*R_{abcd}$, R^*_{abcd} left and, respectively, right duals of the tensor R_{abcd}, page 50
$^+$ Hermitian conjugation, page 96
†, ‡ generalised dualisation operations, page 50
$\sharp$, $^\flat$ musical operators, page 44
${}_\alpha, {}_\beta, {}_\gamma, \ldots$ spatial coordinate indices
${}_{\boldsymbol{A}}, {}_{\boldsymbol{B}}, {}_{\boldsymbol{C}}, \ldots$ spinor frame indices, page 74
${}_{\boldsymbol{a}}, {}_{\boldsymbol{b}}, \ldots$ spacetime frame indices ranging $\mathbf{0}, \ldots, \mathbf{3}$
${}_{\boldsymbol{i}}, {}_{\boldsymbol{j}}, {}_{\boldsymbol{k}}, \ldots$ frame indices ranging either $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ or $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$
$\perp$ perpendicular component
${}_\mu, {}_\nu, {}_\lambda, \ldots$ spacetime coordinate indices
${}_A, {}_B, {}_C, \ldots$ abstract spinor indices, page 66
${}_a, {}_b, {}_c \ldots$ abstract spacetime indices
${}_i, {}_j, {}_k, \ldots$ abstract spatial indices
${}_sY_{lm}$ spin-weighted spherical harmonics
${}_{(a_1 \cdots a_l)}$ symmetrisation over the indices ${}_{a_1 \cdots a_l}$, page 36
${}_{[a_1 \cdots a_l]}$ antisymmetrisation over the indices ${}_{a_1 \cdots a_l}$, page 36
${}_{\mathcal{A}}, {}_{\mathcal{B}}, \ldots$ arbitrary string of indices
${}_{\{a_1 \cdots a_l\}}$ symmetric trace-free part over the indices ${}_{a_1 \cdots a_l}$, page 47