VECTORS, PURE AND APPLIED

A General Introduction to Linear Algebra

Many books on linear algebra focus purely on getting students through exams, but this text explains both the how and the why of linear algebra and enables students to begin thinking like mathematicians. The author demonstrates how different topics (geometry, abstract algebra, numerical analysis, physics) make use of vectors in different ways, and how these ways are connected, preparing students for further work in these areas.

The book is packed with hundreds of exercises ranging from the routine to the challenging. Sketch solutions of the easier exercises are available online.

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In general the position as regards all such new calculi is this. – That one cannot accomplish by them anything that could not be accomplished without them. However, the advantage is, that, provided that such a calculus corresponds to the inmost nature of frequent needs, anyone who masters it thoroughly is able – without the unconscious inspiration which no one can command – to solve the associated problems, even to solve them mechanically in complicated cases in which, without such aid, even genius becomes powerless. . . . Such conceptions unite, as it were, into an organic whole, countless problems which otherwise would remain isolated and require for their separate solution more or less of inventive genius.

(Gauss *Werke*, Bd. 8, p. 298 (quoted by Moritz [24]))

For many purposes of physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing . . . Cartesian coordinates, and to fix the mind at once on a point of space instead of its three coordinates, and on the magnitude and direction of a force instead of its three components. . . . I am convinced that the introduction of the idea [of vectors] will be of great use to us in the study of all parts of our subject, and especially in electrodynamics where we have to deal with a number of physical quantities, the relations of which to each other can be expressed much more simply by [vectorial equations rather] than by the ordinary equations.

(Maxwell *A Treatise on Electricity and Magnetism* [21])

We [Halmos and Kaplansky] share a love of linear algebra. . . . And we share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury.

(Kaplansky in *Paul Halmos: Celebrating Fifty Years of Mathematics* [17])

Marco Polo describes a bridge, stone by stone.

‘But which is the stone that supports the bridge?’ Kublai Khan asks.

‘The bridge is not supported by one stone or another,’ Marco answers, ‘but by the line of the arch that they form.’

Kublai Khan remains silent, reflecting. Then he adds: ‘Why do you speak to me of the stones? It is only the arch that matters to me.’

Polo answers: ‘Without stones there is no arch.’

(Calvino *Invisible Cities* (translated by William Weaver) [8])

Contents

Introduction

There exist several fine books on vectors which achieve concision by only looking at vectors from a single point of view, be it that of algebra, analysis, physics or numerical analysis (see, for example, [18], [19], [23] and [28]). This book is written in the belief that it is helpful for the future mathematician to see all these points of view. It is based on those parts of the first and second year Cambridge courses which deal with vectors (omitting the material on multidimensional calculus and analysis) and contains roughly 60 to 70 hours of lectured material.

The first part of the book contains first year material and the second part contains second year material. Thus concepts reappear in increasingly sophisticated forms. In the first part of the book, the inner product starts as a tool in two and three dimensional geometry and is then extended to $\mathbb{R}^n$ and later to $\mathbb{C}^n$. In the second part, it reappears as an object satisfying certain axioms. I expect my readers to read, or skip, rapidly through familiar material, only settling down to work when they reach new results. The index is provided mainly to help such readers who come upon an unfamiliar term which has been discussed earlier. Where the index gives a page number in a different font (like 389, rather than 389) this refers to an exercise. Sometimes I discuss the relation between the subject of the book and topics from other parts of mathematics. If the reader has not met the topic (morphisms, normal distributions, partial derivatives or whatever), she should simply ignore the discussion.

Random browsers are informed that, in statements involving $\mathbb{F}$, they may take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, that z^* is the complex conjugate of z and that 'self-adjoint' and 'Hermitian' are synonyms. If $T : A \to B$ is a function we sometimes write $T(a)$ and sometimes Ta.

There are two sorts of exercises. The first form part of the text and provide the reader with an opportunity to think about what has just been done. There are sketch solutions to most of these on my home page www.dpmms.cam.ac.uk/∼twk/.

These exercises are intended to be straightforward. If the reader does not wish to attack them, she should simply read through them. If she does attack them, she should remember to state reasons for her answers, whether she is asked to or not. Some of the results are used later, but no harm should come to any reader who simply accepts my word that they are true.

The second type of exercise occurs at the end of each chapter. Some provide extra background, but most are intended to strengthen the reader's ability to use the results of the preceding chapter. If the reader finds all these exercises easy or all of them impossible, she is reading the wrong book. If the reader studies the entire book, there are many more exercises than she needs. If she only studies an individual chapter, she should find sufficiently many to test and reinforce her understanding.

My thanks go to several student readers and two anonymous referees for removing errors and improving the clarity of my exposition. It has been a pleasure to work with Cambridge University Press.

I dedicate this book to the Faculty Board of Mathematics of the University of Cambridge. My reasons for doing this follow in increasing order of importance.

(1) No one else is likely to dedicate a book to it.
(2) No other body could produce Minute 39 (a) of its meeting of 18th February 2010 in which it is laid down that a basis is not an *ordered* set but an *indexed* set.
(3) This book is based on syllabuses approved by the Faculty Board and takes many of its exercises from Cambridge exams.
(4) I need to thank the Faculty Board and everyone else concerned for nearly 50 years spent as student and teacher under its benign rule. Long may it flourish.