Preliminaries

LOWELL W. BEINEKE and ROBIN J. WILSON

1. Graph theory
2. Graph colourings
References

1. Graph theory

This section presents the basic definitions, terminology and notation of graph theory, along with some fundamental results. Further information can be found in the many standard books on the subject – for example, Bondy and Murty [1], Chartrand, Lesniak and Zhang [2], Gross and Yellen [3] and West [5], or, for a simpler treatment, Marcus [4] and Wilson [6].

Graphs

A *graph* G is a pair of sets (V, E), where V is a finite non-empty set of elements called *vertices*, and E is a finite set of elements called *edges*, each of which has two associated vertices. The sets V and E are the *vertex-set* and *edge-set* of G, and are sometimes denoted by $V(G)$ and $E(G)$. The number of vertices in G is called the *order* of G and is usually denoted by n (but sometimes by $|G|$); the number of edges is denoted by m. A graph with only one vertex is called *trivial*.

An edge whose vertices coincide is a *loop*, and if two edges have the same pair of associated vertices, they are called *multiple edges*. In this book, unless otherwise specified, graphs are assumed to have neither loops nor multiple edges; that is, they are taken to be *simple*. Hence, an edge e can be considered as its associated pair of vertices, $e = \{v, w\}$, usually shortened to vw. An example of a graph of order 5 is shown in Fig. 1(a).

The *complement* $\overline{G}$ of a graph G has the same vertices as G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Figure 1(b) shows the complement of the graph in Fig. 1(a).

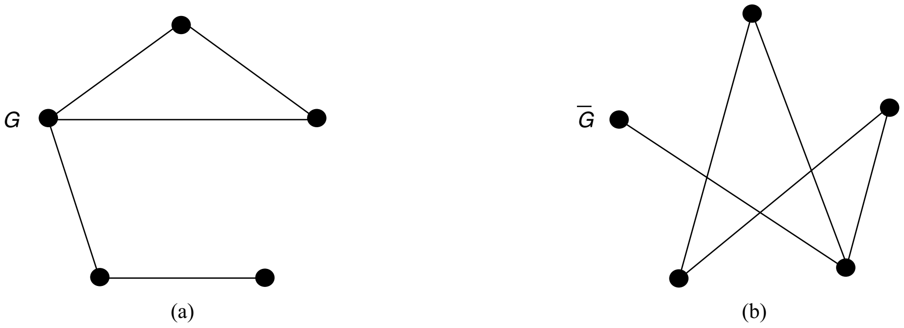

Fig. 1.

Adjacency and degrees

The vertices of an edge are its *endpoints* and the edge is said to *join* these vertices. An endpoint v of an edge $e = vw$ and the edge e are *incident* with each other. Two vertices that are joined by an edge are called *neighbours* and are said to be *adjacent*; if v and w are adjacent vertices we sometimes write $v \sim w$, and if they are not adjacent we write $v \nsim w$. Two edges are *adjacent* if they have a vertex in common.

The set $N(v)$ of neighbours of a vertex v is called its *neighbourhood*. If $X \subseteq V$, then $N(X)$ denotes the set of vertices that are adjacent to some vertex of X.

The *degree* $\deg v$, or $d(v)$, of a vertex v is the number of its neighbours; in a non-simple graph, it is the number of occurrences of the vertex as an endpoint of an edge, with loops counted twice. A vertex of degree 0 is an *isolated vertex* and one of degree 1 is an *end-vertex* or *leaf*. A graph is *regular* if all of its vertices have the same degree, and is *k-regular* if that degree is k; a 3-regular graph is sometimes called *cubic*. The maximum degree in a graph G is denoted by $\Delta(G)$, or just Δ, and the minimum degree by $\delta(G)$ or δ.

An *isomorphism* between two graphs G and H is a bijection between their vertex-sets that preserves both adjacency and non-adjacency. The graphs G and H are *isomorphic*, written $G \cong H$, if there exists an isomorphism between them.

Independent sets and cliques

A set of vertices of a graph G is an *independent set* (or *stable set*) if no two vertices are adjacent. The *independence number* (or *stability number*) $\alpha(G)$ is the size of the largest such set.

A set of vertices is a *clique* if all pairs of vertices are adjacent. The *clique number* $\omega(G)$ is the size of a largest clique.

Walks, paths and cycles

A *walk* in a graph is a sequence of vertices and edges $v_0, e_1, v_1, \ldots, e_k, v_k$, in which the edge e_i joins the vertices v_{i-1} and v_i. This walk is said to *go from v_0 to v_k* or to *connect* v_0 *and* v_k, and is called a v_0–v_k walk. It is frequently shortened to $v_0 v_1 \cdots v_k$, since the edges can be inferred from this. A walk is *closed* if the first and last vertices are the same. Some important types of walk are the following:

- a *path* is a walk in which no vertex is repeated
- a *cycle* is a non-trivial closed walk in which no vertex is repeated, except the first and last
- a *trail* is a walk in which no edge is repeated
- a *circuit* is a non-trivial closed trail.

Connectedness and distance

A graph is *connected* if it has a path connecting each pair of vertices, and *disconnected* otherwise. A (*connected*) *component* of a graph is a maximal connected subgraph.

The number of occurrences of edges in a walk is called its *length*, and in a connected graph the *distance* $d(v, w)$ from v to w is the length of a shortest v–w path. It is easy to check that distance satisfies the properties of a metric. The *diameter* of a connected graph G is the greatest distance between any pair of vertices in G. If G has a cycle, the *girth* of G is the length of a shortest cycle.

A connected graph is *Eulerian* if it has a closed trail containing all of its edges; such a trail is an *Eulerian trail*. A connected graph G is Eulerian if and only if every vertex of G has even degree. This means that the edge-set of G can be partitioned into cycles.

A graph of order n is *Hamiltonian* if it has a cycle of length n, and is *pancyclic* if it has a cycle of every length from 3 to n. It is *traceable* if it has a path through all vertices. No 'good' characterizations of these properties are known.

Bipartite graphs and trees

If the set of vertices of a graph G can be partitioned into two non-empty subsets so that no edge joins two vertices in the same subset, then G is *bipartite*. The two subsets are called *partite sets*, and if they have orders r and s, G is said to be an $r \times s$ *bipartite graph*. (For convenience, the graph with one vertex and no edges is also called bipartite.) Bipartite graphs are characterized by having no cycles of odd length.

Among the bipartite graphs are *trees*, those connected graphs with no cycles. Any graph without cycles is a *forest*; thus, each component of a forest is a tree. Trees have been characterized in many ways, some of which we give here. For a graph G of order n, the following statements are equivalent:

- G is connected and has no cycles
- G is connected and has $n - 1$ edges
- G has no cycles and has $n - 1$ edges
- G has exactly one path between any two vertices.

The set of trees can also be defined inductively: a single vertex is a tree; and for $n \geq 1$, the trees with $n+1$ vertices are those graphs obtainable from some tree with n vertices by adding a new vertex adjacent to precisely one of its vertices.

This definition has a natural extension to higher dimensions. The *k-dimensional trees*, or *k-trees* for short, are defined as follows: the complete graph on k vertices is a k-tree, and for $n \geq k$, the k-trees with $n+1$ vertices are those graphs obtainable from some k-tree with n vertices by adding a new vertex adjacent to k mutually adjacent vertices in the k-tree. Figure 2 shows a tree and a 2-tree.

Fig. 2.

An important concept in the study of graph minors (introduced later) is the *tree-width* of a graph G, the minimum dimension of any k-tree that contains G as a subgraph.

Special graphs

We now introduce some individual types of graph:

- the *complete graph* K_n has n vertices, each adjacent to all the others
- the *null graph* $\overline{K}_n$ has n vertices and no edges
- the *path graph* P_n consists of the vertices and edges of a path of length $n-1$
- the *cycle graph* C_n consists of the vertices and edges of a cycle of length n
- the *complete bipartite graph* $K_{r,s}$ is the $r \times s$ bipartite graph in which each vertex is adjacent to all of the vertices in the other partite set
- the *complete k-partite graph* $K_{r_1,r_2,\ldots,r_k}$ has its vertices in k sets with orders $r_1, r_2, \ldots, r_k$, and every vertex is adjacent to all of the vertices in the other sets; if the k sets all have order r, the graph is denoted by $K_{k(r)}$.

Examples of these graphs are given in Fig. 3.

Operations on graphs

Let G and H be graphs with disjoint vertex-sets $V(G) = \{v_1, v_2, \ldots, v_r\}$ and $V(H) = \{w_1, w_2, \ldots, w_s\}$.

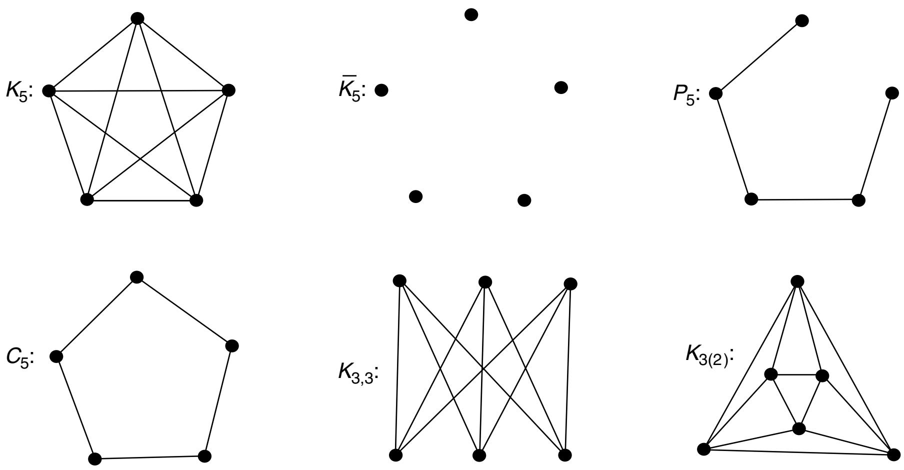

Fig. 3.

The *union* $G \cup H$ has vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. The union of k graphs isomorphic to G is denoted by kG.

The *join* $G + H$ is obtained from $G \cup H$ by adding an edge from each vertex in G to each vertex in H.

The *Cartesian product* $G \times H$ (or $G \,\square\, H$) has vertex-set $V(G) \times V(H)$, with (v_i, w_j) adjacent to (v_h, w_k) if either v_i is adjacent to v_h in G and $w_j = w_k$, or $v_i = v_h$ and w_j is adjacent to w_k in H; in less formal terms, $G \times H$ can be obtained by taking n copies of H and joining corresponding vertices in different copies whenever there is an edge in G.

Examples of these binary operations are given in Fig. 4.

Subgraphs and minors

If G and H are graphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G, and is a *spanning subgraph* if $V(H) = V(G)$. The subgraph $\langle S \rangle$ (or $G[S]$) *induced* by a non-empty set of S of vertices of G is the subgraph H whose vertex-set is S and whose edge-set consists of those edges of G that join two vertices in S. A subgraph H of G is called an *induced subgraph* if $H = \langle V(H) \rangle$. In Fig. 5, H_1 is a spanning subgraph of G, and H_2 is an induced subgraph.

The *deletion of a vertex v* from a graph G results in the subgraph obtained by removing v and all of its incident edges; it is denoted by $G - v$ and is the subgraph induced by $V - \{v\}$. More generally, if S is any set of vertices in G, then $G - S$ is the graph obtained from G by deleting all of the vertices in S and their incident edges; that is, $G - S = \langle V(G) - S \rangle$. Similarly, the *deletion of an edge e* results in the subgraph $G - e$ and, for any set X of edges, $G - X$ is the graph obtained from G by deleting all the edges in X.

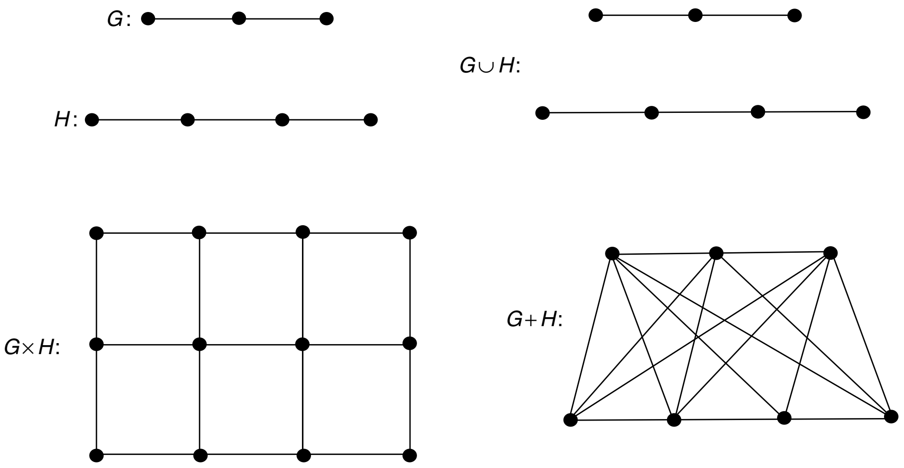

Fig. 4.

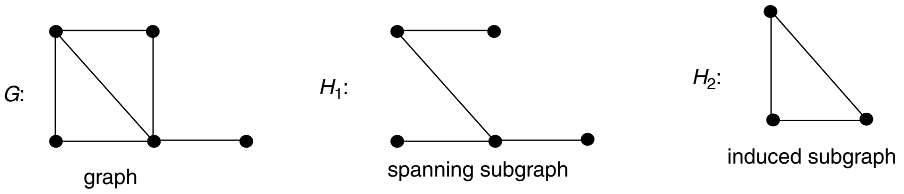

Fig. 5.

If the edge e joins vertices v and w, then the *subdivision* of e replaces e by a new vertex u and two new edges vu and uw. Two graphs are *homeomorphic* if there is some graph from which each can be obtained by a sequence of subdivisions. The *contraction* of e replaces its vertices v and w by a new vertex u, with an edge ux if v or w is adjacent to x in G. The operations of subdivision and contraction are illustrated in Fig. 6.

If H can be obtained from G by a sequence of edge-contractions and the removal of isolated vertices, then G is *contractible* to H. A *minor* of G is any graph that can be obtained from G by a sequence of edge-deletions and edge-contractions, along with deletions of isolated vertices. Note that if G has a subgraph homeomorphic to H, then H is a minor of G.

Digraphs

Digraphs are directed analogues of graphs, and thus have many similarities, as well as some important differences. A *digraph* (or *directed graph*) D is a pair of sets (V, A), where V is a finite non-empty set of elements called *vertices*, and A is a set of ordered pairs of distinct elements of V called *arcs*. Note that the elements of A are ordered,

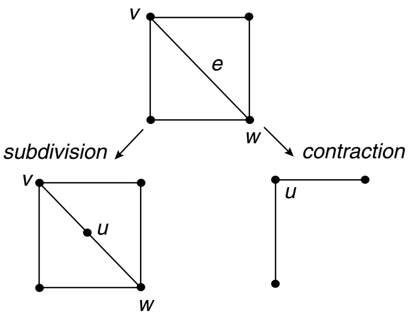

Fig. 6.

which gives each of them a direction. An example of a digraph, with the directions indicated by arrows, is shown in Fig. 7.

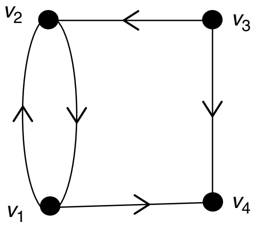

Fig. 7.

Because of the similarities between graphs and digraphs, we mention only the main differences here and do not redefine those concepts that carry over easily. An arc (v, w) in a digraph may be written as vw, and is said to *go from v to w*, or to *go out of v* and *into w*. In the context of digraphs, walks, paths, cycles, trails and circuits are all understood to be directed, unless otherwise indicated. A digraph D is *strongly connected* or *strong* if there is a path from each vertex to each of the others; note that the digraph in Fig. 7 is strong. A *strong component* is a maximal strongly connected subgraph. Every vertex is in at least one strong component, and an edge is in a strong component if and only if it is on a directed cycle.

The *out-degree* $d^+(v)$ of a vertex v in a digraph D is the number of arcs out of v, and the *in-degree* $d^-(v)$ is the number of arcs into v. The minimum out-degree in a digraph is denoted by δ^+, the minimum in-degree δ^-, and the minimum of the two is denoted by δ.

Connectivity

In this section, we give the primary definitions and some of the basic results on connectivity, including two versions of the most important one of all, Menger's theorem.

A vertex v in a graph G is a *cut-vertex* if $G - v$ has more components than G. For a connected graph, this is equivalent to saying that $G - v$ is disconnected, and that there exist vertices u and w, different from v, for which v is on every u–w path. It is easy to see that every non-trivial graph has at least two vertices that are not cut-vertices.

A non-trivial graph is *non-separable* if it is connected and has no cut-vertices. Note that under this definition the graph K_2 is non-separable. There are many characterizations of the other non-separable graphs, as the following statements are all equivalent for a connected graph G with at least three vertices:

- G is non-separable
- every two vertices of G share a cycle
- every two edges of G share a cycle
- for any three vertices u, v and w in G, there is a v–w path that contains u
- for any three vertices u, v and w in G, there is a v–w path that does not contain u.

A *block* in a graph is a maximal non-separable subgraph. Each edge of a graph lies in exactly one block, while each vertex that is not an isolated vertex lies in at least one block, those that are in more than one block being cut-vertices. The graph in Fig. 8 has four blocks.

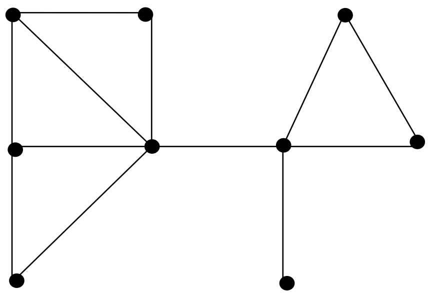

Fig. 8.

The basic idea of non-separability has a natural generalization: a graph G is k*-connected* if the removal of fewer than k vertices always leaves a non-trivial connected graph. The main result on graph connectivity – indeed, it might well be called the Fundamental theorem of connectivity – is Menger's theorem, first published in 1927. It has many equivalent forms, and the first that we give here is the vertex version. Paths joining the same pair of vertices are called *internally disjoint* if they have no other vertices in common.

Menger's theorem (vertex version) *A graph is k-connected if and only if every pair of vertices are joined by k internally disjoint paths.*

The *connectivity* $\kappa(G)$ of a graph G is the maximum non-negative integer k for which G is k-connected; for example, the connectivity of the complete graph K_n is $n - 1$, and a graph has connectivity 0 if and only if it is trivial or disconnected.

There is an analogous body of material that involves edges rather than vertices, and because of the similarities, we treat it in less detail.

An edge e is a *cut-edge* (or *bridge*) of a graph G if $G-e$ has more components than G. (In contrast to the situation with vertices, the removal of an edge cannot increase the number of components by more than 1.) An edge e is a cut-edge if and only if there exist vertices v and w for which e is on every v–w path. The cut-edges in a graph are also characterized by the property of not lying on a cycle; thus, a graph is a forest if and only if every edge is a cut-edge. Graphs with no cut-edges can be characterized in a variety of ways similar to those having no cut-vertices – that is, non-separable graphs. The concepts corresponding to cycles and paths for vertices are circuits and trails for edges.

Moving beyond cut-edges, we have the following definitions. A graph G is *l-edge-connected* if the removal of fewer than l edges always leaves a connected graph. Here is another version of Menger's theorem.

Menger's theorem (edge version) *A graph is l-edge-connected if and only if each pair of its vertices is joined by l edge-disjoint paths.*

The *edge-connectivity* $\lambda(G)$ of a graph G is the greatest non-negative integer l for which G is l-edge-connected. Obviously, $\lambda(G)$ cannot exceed the minimum degree of a vertex of G; furthermore, it is at least as large as the connectivity – that is,

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Along with the undirected versions of Menger's theorem, there are corresponding directed versions (with directed paths and strong connectivity) and weighted versions.

2. Graph colourings

The origins of chromatic graph theory lie in the colouring of maps, a story that is well known. In this section we present some of the definitions and basic results of chromatic graph theory.

Vertex-colourings

A *colouring* of a graph G is an assignment of a colour to each vertex of G so that adjacent vertices always have different colours, and G is said to be *k-colourable* if it has a colouring with k colours. The *chromatic number* $\chi(G)$ is the smallest value of k for which G has a k-colouring.

The fact that computing the chromatic number of a graph is an NP-complete problem has contributed to the attraction of this area of mathematics – in fact, determining whether a graph is 3-colourable is itself NP-complete.

We note that the complete graph K_n has chromatic number n and that a bipartite graph with at least one edge has chromatic number 2, and consequently $\chi(G) \geq 3$ if and only if G contains an odd cycle of odd length. An interesting and useful upper bound on the chromatic number of graphs in general was published by L. Brooks in 1941.

Theorem 2.1 *If G is a connected graph with maximum degree* Δ*, then*

$$\chi(G) \leq \Delta + 1,$$

with equality if and only if G is a complete graph or a cycle of odd length.

There is also an upper bound on the chromatic number that is in terms of minimum degrees; it is easily proved by induction.

Theorem 2.2 *If G and each of its subgraphs has a vertex of degree* δ^* *or less, then*

$$\chi(G) \leq \delta^* + 1.$$

As far as lower bounds are concerned, the obvious one is the clique number: $\chi(G) \geq \omega(G)$. However, as was first shown by Blanche Descartes, there are triangle-free graphs with arbitrarily large chromatic numbers. More generally, Paul Erdős proved the following result.

Theorem 2.3 *For all* $k \geq 2$ *and all g, there exists a k-chromatic graph with girth greater than g.*

We conclude our discussion of bounds with a pair that involve the independence number, the dual concept to the clique number.

Theorem 2.4 *If G is a graph of order n and independence number* α*, then*

$$\frac{n}{\alpha} \leq \chi(G) \leq n - \alpha + 1.$$

Critical graphs

In the study of the chromatic number, one type of graph that arises quite naturally is that of a *critical graph*, a graph G for which each proper subgraph has chromatic number less than that of G. If $\chi(G) = k$, they are often called *k-critical*; they were first studied by G. A. Dirac. Here are two of his results.

Theorem 2.5 *For* $k \geq 2$*, every k-critical graph is* $(k-1)$*-edge-connected.*

Theorem 2.6 *Every k-critical graph is either Hamiltonian or has a cycle of length at least* $2k - 2$*.*