Gravity

This textbook explores approximate solutions to general relativity and their consequences. It offers a unique presentation of Einstein's theory by developing powerful methods that can be applied to astrophysical systems.

Beginning with a uniquely thorough treatment of Newtonian gravity, the book develops post-Newtonian and post-Minkowskian approximation methods to obtain weak-field solutions to the Einstein field equations. The book explores the motion of self-gravitating bodies, the physics of gravitational waves, and the impact of radiative losses on gravitating systems. It concludes with a brief overview of alternative theories of gravity.

Ideal for graduate courses on general relativity and relativistic astrophysics, the book examines real-life applications, such as planetary motion around the Sun, the timing of binary pulsars, and gravitational waves emitted by binary black holes. Text boxes explore related topics and provide historical context, and over 100 exercises present challenging tests of the material covered in the main text.

Eric Poisson is Professor of Physics at the University of Guelph. He is a Fellow of the American Physical Society and serves on the Editorial Boards of *Physical Review Letters* and *Classical and Quantum Gravity*.

Clifford M. Will is Distinguished Professor of Physics at the University of Florida and J. S. McDonnell Professor Emeritus at Washington University in St. Louis. He is a member of the US National Academy of Sciences, and Editor-in-Chief of *Classical and Quantum Gravity*. He is well known for his ability to bring science to broad audiences.

Gravity

Newtonian, Post-Newtonian, Relativistic

ERIC POISSON University of Guelph

CLIFFORD M. WILL University of Florida





University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is a part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org Information on this title: www.cambridge.org/9781107032866

© E. Poisson and C. Will 2014

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

> First published 2014 Reprinted 2014

Printed in the United Kingdom by TJ International Ltd, Padstow, Cornwall

A catalog record for this publication is available from the British Library

ISBN 978-1-107-03286-6 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Contents

| List of boxes | | <i>page</i> viii |
|---------------|--|------------------|
| Preface | 2 | xi |
| | | |
| 1 Four | 1 | |
| 1.1 | Newtonian gravity | 1 |
| 1.2 | Equations of Newtonian gravity | 3 |
| 1.3 | Newtonian field equation | 7 |
| 1.4 | Equations of hydrodynamics | 10 |
| 1.5 | Spherical and nearly spherical bodies | 27 |
| 1.6 | Motion of extended fluid bodies | 45 |
| 1.7 | Bibliographical notes | 60 |
| 1.8 | Exercises | 61 |
| 2 Stru | cture of self-gravitating bodies | 63 |
| 2.1 | Equations of internal structure | 64 |
| 2.2 | Equilibrium structure of a spherical body | 66 |
| 2.3 | Rotating self-gravitating bodies | 89 |
| 2.4 | General theory of deformed bodies | 105 |
| 2.5 | Tidally deformed bodies | 119 |
| 2.6 | Bibliographical notes | 135 |
| 2.7 | Exercises | 135 |
| 2 No | | 120 |
| 3 New | | 138 |
| 3.1 | Celestial mechanics from Newton to Einstein | 138 |
| 3.2 | Iwo bodies: Kepler's problem | 140 |
| 3.3 | Perturbed Kepler problem | 154 |
| 5.4 2.5 | Case studies of perturbed Repletian motion | 101 |
| 3.3 2.6 | More bodies | 1/3 |
| 5.0 2.7 | Lagrangian formulation of Newtonian dynamics | 181 |
| 3./ 2.0 | Biolographical notes | 184 |
| 3.8 | Exercises | 185 |
| 4 Minl | kowski spacetime | 189 |
| 4.1 | Spacetime | 189 |
| 4.2 | Relativistic hydrodynamics | 203 |
| 4.3 | Electrodynamics | 208 |

v

| vi | Contents | | | | |
|----|--------------------|---|-----|--|--|
| | | | | | |
| | 4.4 | Point particles in spacetime | 211 | | |
| | 4.5 | Bibliographical notes | 214 | | |
| | 4.6 | Exercises | 214 | | |
| | 5 Curved spacetime | | | | |
| | 5.1 | Gravitation as curved spacetime | 217 | | |
| | 5.2 | Mathematics of curved spacetime | 225 | | |
| | 5.3 | Physics in curved spacetime | 243 | | |
| | 5.4 | Einstein field equations | 250 | | |
| | 5.5 | Linearized theory | 252 | | |
| | 5.6 | Spherical bodies and Schwarzschild spacetime | 264 | | |
| | 5.7 | Bibliographical notes | 284 | | |
| | 5.8 | Exercises | 285 | | |
| | 6 Post | -Minkowskian theory: Formulation | 290 | | |
| | 6.1 | Landau-Lifshitz formulation of general relativity | 291 | | |
| | 6.2 | Relaxed Einstein equations | 301 | | |
| | 6.3 | Integration of the wave equation | 308 | | |
| | 6.4 | Bibliographical notes | 325 | | |
| | 6.5 | Exercises | 326 | | |
| | 7 Post | -Minkowskian theory: Implementation | 328 | | |
| | 7.1 | Assembling the tools | 329 | | |
| | 7.2 | First iteration | 341 | | |
| | 7.3 | Second iteration: Near zone | 344 | | |
| | 7.4 | Second iteration: Wave zone | 361 | | |
| | 7.5 | Bibliographical notes | 365 | | |
| | 7.6 | Exercises | 366 | | |
| | 8 Post- | -Newtonian theory: Fundamentals | 371 | | |
| | 8.1 | Equations of post-Newtonian theory | 371 | | |
| | 8.2 | Classic approach to post-Newtonian theory | 378 | | |
| | 8.3 | Coordinate transformations | 381 | | |
| | 8.4 | Post-Newtonian hydrodynamics | 400 | | |
| | 8.5 | Bibliographical notes | 410 | | |
| | 8.6 | Exercises | 410 | | |
| | 9 Post- | 414 | | | |
| | 9.1 | From fluid configurations to isolated bodies | 414 | | |
| | 9.2 | Inter-body metric | 423 | | |
| | 9.3 | Motion of isolated bodies | 431 | | |
| | 9.4 | Motion of compact bodies | 445 | | |
| | 9.5 | Motion of spinning bodies | 454 | | |
| | 9.6 | Point particles | 474 | | |

| vii | Contents | | | |
|-----|------------|--|----------|--|
| | | | | |
| | 9.7 | Bibliographical notes | 47 | |
| | 9.8 | Exercises | 47 | |
| | 10 Post-N | lewtonian celestial mechanics, astrometry and navigation | 48 | |
| | 10.1 | Post-Newtonian two-body problem | 48 | |
| | 10.2 | Motion of light in post-Newtonian gravity | 49 | |
| | 10.3 | Post-Newtonian gravity in timekeeping and navigation | 50 | |
| | 10.4 | Spinning bodies | 52 | |
| | 10.5 | Bibliographical notes | 53 | |
| | 10.6 | Exercises | 5. | |
| | 11 Gravita | ational waves | 5. | |
| | 11.1 | Gravitational-wave field and polarizations | 54 | |
| | 11.2 | The quadrupole formula | 5: | |
| | 11.3 | Beyond the quadrupole formula: Waves at 1.5PN order | 50 | |
| | 11.4 | Gravitational waves emitted by a two-body system | 6 | |
| | 11.5 | Gravitational waves and laser interferometers | 6 | |
| | 11.6 | Bibliographical notes | 6 | |
| | 11.7 | Exercises | 6 | |
| | 12 Radiat | ive losses and radiation reaction | 6 | |
| | 12.1 | Radiation reaction in electromagnetism | 62 | |
| | 12.2 | Radiative losses in gravitating systems | 6. | |
| | 12.3 | Radiative losses in slowly-moving systems | 64 | |
| | 12.4 | Astrophysical implications of radiative losses | 6 | |
| | 12.5 | Radiation-reaction potentials | 63 | |
| | 12.6 | Radiation reaction of fluid systems | 6 | |
| | 12.7 | Radiation reaction of N-body systems | 6 | |
| | 12.8 | Radiation reaction in alternative gauges | 6 | |
| | 12.9 | Orbital evolution under radiation reaction | 6 | |
| | 12.10 | Bibliographical notes | 6 | |
| | 12.11 | Exercises | 6 | |
| | 13 Altern | ative theories of gravity | 6 | |
| | 13.1 | Metric theories and the strong equivalence principle | 70 | |
| | 13.2 | Parameterized post-Newtonian framework | 70 | |
| | 13.3 | Experimental tests of gravitational theories | 72 | |
| | 13.4 | Gravitational radiation in alternative theories of gravity | 73 | |
| | 13.5 | Scalar–tensor gravity | 72 | |
| | 13.6 | Bibliographical notes | 7: | |
| | 13.7 | Exercises | 7. | |
| | Reference | es | 7 | |
| | 1 1 | | יר יד | |

Boxes

| 1.1 Tests of the weak equivalence principle | 4 |
|--|-----|
| 1.2 Proof that $\nabla^2 \mathbf{x} - \mathbf{x}' ^{-1} = -4\pi \delta(\mathbf{x} - \mathbf{x}')$ | 9 |
| 1.3 Proof that $\mathcal{V}^{-1}d\mathcal{V}/dt = \nabla \cdot \boldsymbol{v}$ | 12 |
| 1.4 Symmetrized and antisymmetrized indices | 24 |
| 1.5 Spherical harmonics | 32 |
| 1.6 Proof that $r^2 \nabla^2 n^{\langle L \rangle} = -\ell(\ell+1)n^{\langle L \rangle}$ | 44 |
| 1.7 Is the center-of-mass unique? | 48 |
| 2.1 Newtonian gravity, neutrinos, and the Sun | 69 |
| 2.2 Integration of the Lane–Emden equation | 75 |
| 2.3 Clairaut-Radau equation and Love numbers | 112 |
| 2.4 Driven harmonic oscillator | 124 |
| 3.1 Solving Kepler's equation | 149 |
| 3.2 Orbital and fundamental frames | 153 |
| 3.3 Variation of arbitrary constants | 156 |
| 3.4 DI Herculis: A tidal troublemaker | 171 |
| 4.1 Tests of special relativity | 191 |
| 4.2 Relativistic mass | 196 |
| 4.3 Photons: An alternative viewpoint | 199 |
| 5.1 Uniform gravitational fields | 221 |
| 5.2 Vector calculus in polar coordinates | 228 |
| 5.3 Riemann normal coordinates | 238 |
| 5.4 Fermi normal coordinates | 241 |
| 5.5 Hydrostatic equilibrium | 246 |
| 5.6 Geometric optics | 248 |
| 5.7 Decomposition of vectors and tensors into irreducible pieces | 257 |
| 5.8 Birkhoff's theorem in Newtonian gravity | 267 |
| 5.9 Neutron stars | 283 |
| 6.1 Two versions of energy-momentum conservation | 293 |
| 6.2 Existence of harmonic coordinates | 302 |
| 6.3 Wave equation in flat and curved spacetimes | 303 |
| 6.4 The expansion parameter G | 305 |
| 6.5 Green's function for the wave equation | 309 |
| 6.6 Dipole solution to the wave equation | 312 |
| 6.7 Solution to the wave equation | 324 |
| 7.1 Radiation-reaction terms in the potentials | 335 |
| 7.2 Multipole structure of the wave-zone metric | 340 |

viii

| ix | List of boxes | | | |
|----|--|-----|--|--|
| | | | | |
| | 7.3 Definition of the superpotential | 352 | | |
| | 7.4 Three-point function $K(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2)$ | 356 | | |
| | 7.5 Near-zone potentials | 358 | | |
| | 7.6 Post-Minkowskian theory and the slow-motion approximation | 360 | | |
| | 7.7 Wave-zone fields | 364 | | |
| | 8.1 Maxwell-like formulation of post-Newtonian theory | 376 | | |
| | 8.2 Post-Newtonian transformations | 382 | | |
| | 8.3 Rotating coordinates | 385 | | |
| | 8.4 Integration and time differentiation | 404 | | |
| 1 | 0.1 Ambiguities in energy and angular momentum | 483 | | |
| 1 | 0.2 Spherical trigonometry | 498 | | |
| 1 | 0.3 The "Newtonian" deflection of light | 500 | | |
| 1 | 0.4 Global Positioning System | 517 | | |
| 1 | 1.1 Why 45 degrees? | 550 | | |
| 1 | 1.2 The quadrupole-formula controversy | 553 | | |
| 1 | 1.3 Field integrals | 574 | | |
| 1 | 1.4 Gravitational-wave field to 1.5PN order | 600 | | |
| 1 | 2.1 Redefining the energy | 632 | | |
| 1 | 2.2 Momentum flux and gravitational-wave beaming | 648 | | |
| 1 | 2.3 Radiation-reaction potentials | 664 | | |
| 1 | 2.4 Multi-scale analysis | 686 | | |
| 1 | 3.1 Parameterized post-Newtonian metric | 704 | | |
| 1 | 3.2 Lunar laser ranging and the Nordtvedt effect | 729 | | |
| 1 | 3.3 Distortion of a ring of particles by a gravitational wave | 737 | | |
| 1 | 3.4 Nordtvedt effect and the variation of $G_{\rm eff}$ | 749 | | |

Preface

During the past forty years or so, spanning roughly our careers as teachers and research scientists, Einstein's theory of general relativity has made the transition from a largely mathematical curiosity with limited relevance to the real world to arguably the centerpiece of our effort to understand the universe on all scales.

At the largest scales, those of the universe as a whole, cosmology and general relativity are joined at the hip. You can't do one without the other. At the smallest scales, those of the Planck time, Planck length, and Planck energy, general relativity and particle physics are joined at the hip. String theory, loop quantum gravity, the multiverse, branes and bulk – these are arenas where the geometry of Einstein and the physics of the quantum may be inextricably linked. These days it seems that you can't do one without the other.

At the intermediate scales that interest astronomers, general relativity and astrophysics are becoming increasingly linked. You can still do one without the other, but it's becoming harder. One of us is old enough to remember a time when the majority of astronomers felt that black holes would never amount to much, and that it was a waste of time to worry about general relativity. Today black holes and neutron stars are everywhere in the astronomy literature, and gravitational lensing – the tool that relies on the relativistic bending of light – is used for everything from measuring dark energy to detecting exoplanets.

Given the surge of interest in general relativity, it is no surprise that the last several years have witnessed the publication of a multitude of new textbooks on Einstein's theory. Many of them are cut from a very similar cloth: they cover the fundamentals of the theory at an introductory level, including the spacetime formulation of special relativity, elements of differential geometry, the Einstein field equations, black holes, gravitational waves, and cosmology. This book is cut from a very different cloth. Here you will not (spoiler alert!) find any discussion of cosmology, and although black holes will appear in many places, you will not find anything about the joys and wonders of the Kerr metric.

This book is about *approximations* to Einstein's theory of general relativity, and their applications to planetary motion around the Sun, to the timing of binary pulsars, to gravitational waves emitted by binary black holes, and to many other real-life, astrophysical systems.

The first approximation to general relativity is, of course, Newton's gravity. Although the theories are conceptually very different, it must be admitted that the overwhelming majority of phenomena in the universe can be very adequately described by the laws of Newtonian gravity. To a high degree of accuracy, Newton rules the Sun, the Earth, the solar system, all normal stars, galaxies, and clusters of galaxies. Accordingly, almost a quarter of this book is devoted to Newton's theory. This choice reflects one of our (not so) hidden agendas. During our careers of teaching general relativity and advising graduate students, we have

xi

xii

Preface

too often encountered students who are superbly motivated to study Einstein's theory, but who cannot say more than "inverse square law" and "elliptical orbits" when asked what they know about Newtonian gravity. In our view, general relativity is a theory of gravity, and if you wish to comprehend its importance for astrophysics, you must first master what Newton has to say about gravitating bodies, rotating bodies, tidally interacting bodies, perturbed Keplerian orbits, and so on. We therefore make it our mission, in Chapters 1, 2, and 3, to provide a thorough discussion of the wonders of Newtonian gravity.

In the following two chapters we quickly review special relativity, the foundations of general relativity as a metric theory of gravity, the mathematical formulation of the theory, and its most famous solution, the Schwarzschild metric. We emphasize that Chapters 4 and 5 are very much a minimal package. The coverage is sufficient for our intended purposes in the remainder of the book, but it is no substitute for a proper education in general relativity that can be acquired from the traditional textbooks.

We get to our main point by Chapter 6. This is the development of a set of systematic schemes, known as post-Minkowskian theory and post-Newtonian theory, for obtaining approximate solutions to the Einstein field equations. The idea is to go from the exact theory, which governs the behavior of arbitrarily strong fields, such as those near black holes, to a useful approximation that applies to weak fields, such as those inside and near the Sun, those inside and near white dwarfs, and those at a safe distance from neutron stars and black holes. The approximation, of course, reproduces the predictions of Newtonian theory, but we go beyond this and formulate a method of approximation that can be pushed systematically to higher and higher order, and generate increasingly accurate descriptions of a weak gravitational field. Along the way, we make the case that this approximation can also describe important situations involving compact objects such as neutron stars and black holes; not the up-close-and-personal geometry of a compact object, to be sure, but its motion around another body (compact or not), so long as the mutual gravitational attraction is weak.

This program occupies us through Chapters 6, 7, 8, and 9. In Chapter 10 we apply the approximation methods to the description of relativistic effects on the dynamics of the solar system, the measurement of time on the Earth's surface and in orbit, the bending of light by a massive body, and the dynamics of spinning bodies. In Chapter 11 we explore the rich physics of gravitational waves, and in Chapter 12 we investigate the impact of radiative losses on the dynamics of gravitating systems. We conclude the book in Chapter 13 with a brief overview of alternative theories of gravity.

The central theme of this book is therefore the physics of weak gravitational fields. The reader may object that we give up too much by eliminating strong fields from our discussion; after all, exact solutions to the Einstein field equations describe the full richness of curved spacetime, whether strong or weak. Unfortunately, there are *extremely few* exact solutions to Einstein's equations that are physically interesting. The Schwarzschild solution is obviously interesting and important, and so is the Kerr solution for rotating black holes (although the Kerr metric makes no appearance in this book). But no exact solution to Einstein's equations has ever been found that describes a simple double-star system in orbital motion. And no exact solution is known that describes any kind of bounded, physical system that radiates gravitational waves.

xiii

Preface

The problem is that Einstein's field equations are so complicated that it is almost always necessary to impose a high degree of symmetry (spherical symmetry, spatial homogeneity, stationarity, etc.) in order to make progress toward finding a solution. Furthermore, a solution to Einstein's equations is, by definition, a spacetime; it must encompass the entire past history and future fate of the system, everywhere in space. For a binary-star system, for example, the solution must, at least in principle, run from the distant past, when a tenuous cloud of gas coalesced to form the stars, all the way to the distant future, when the stars, having possibly collapsed to form neutron stars or black holes along the way, have merged into a single object (possibly a single black hole); it must also describe the gravitational waves that are generated during the entire time by the orbital motion and merger of the two stars, and by the relaxation of the merged object to a final stationary state. It should not come as a surprise that nobody has found a solution that describes such a wide range of phenomena. Ironically, a body of beautiful mathematical work has demonstrated conclusively that given suitable initial conditions, a solution to Einstein's equations always exists, at least within a specified part of the spacetime. Sadly, such existence theorems do not tell us how to find such solutions.

Often, when one talks about exact solutions to the Einstein field equations, one means analytic solutions, or solutions that can be expressed in terms of reasonably well-known mathematical functions. Perhaps this is too restrictive. What about numerical solutions? Given a sufficiently powerful computer, it should be possible to solve Einstein's equations numerically without imposing any symmetries. After all, the field equations of general relativity are partial differential equations, and these can readily be converted into the kind of difference equations that are suited to digital computing. This has turned out to be a very difficult challenge. Part of the difficulty is computational: simulation of the simplest spacetimes requires enormous computational power and memory. Part of the difficulty is mathematical: one must identify, from a broad spectrum of possibilities, a formulation of the field equations that is best suited for numerical work. There has been enormous progress on these fronts in the last 20 years, and spectacular breakthroughs have occurred in the last ten. Today (in 2013), numerical relativity is a major sub-branch of gravitational physics. It is now possible to simulate the final dozen orbits of two inspiralling and merging compact objects (black holes or neutron stars), the gravitational collapse of a dead stellar core on its way to form a supernova, the formation and evolution of accretion disks around black holes, the interaction of a binary neutron-star system with the strong magnetic fields it supports, and the generation of gravitational waves by such strongly gravitating systems.

As spectacular as this progress has been, at present it is still not possible to simulate the final thousand orbits of a compact binary inspiral. The limitations are both technical (a vast range of grid resolutions is required) and computational (insufficient memory and speed, even with the largest parallel processors). But approximately 990 of those orbits can be described by the weak-field methods that we develop in this book. It was found that there is a very good agreement between the approximation methods and those of numerical relativity when their domains of applicability overlap. So in addition to their obvious applications to the solar system, the weak-field methods have proved to be unreasonably effective in describing situations, such as the late stages of binary inspirals, where the fields are not so

xiv

Preface

weak and the motions not so slow. And the combination of these methods with numerical relativity has proved to be a powerful tool for many important problems.

The vast majority of high-precision experiments that were carried out to test general relativity can be fully understood on the basis of the post-Newtonian methods that we develop in this book. And even though the departures from Newtonian gravity are very, very small on and around Earth, modern technology has made them not only detectable, but also *essentially important* in the precision measurement of time. A well-known example is the Global Positioning System, which simply would not work if relativistic corrections were not taken into account. Today every relativist proudly points to the GPS as an example – admittedly, perhaps, the only example – of a practical application of general relativity. We describe how this comes about in Chapter 10.

Finally, a central motivation for this book is the expectation that soon after its initial publication, gravitational waves will be measured directly and routinely, and that gravitationalwave astronomy, enabled by ground-based laser interferometers, by pulsar timing arrays, and possibly by a future space-based antenna, will become a new standard way of "listening" to the universe. The approximation methods that we develop in this book are *the* tools for understanding gravitational radiation, and it is our hope that students and researchers wishing to join this new scientific venture will turn to our book to learn and master these tools.

Acknowledgments

We would like to acknowledge colleagues and students who contributed important comments and corrections during the writing of this book: Emanuele Berti, Ryan Lang, Saeed Mirshekari, Laleh Sadeghian, Nico Yunes, and Ian Vega.

CMW is grateful to Washington University in St. Louis for its support during the early phase of writing, particularly during a sabbatical leave in 2010–2011. He also thanks the Institut d'Astrophysique de Paris for its hospitality during this sabbatical, and during extended stays in 2009, 2012, and 2013. Finally he is grateful to the US National Science Foundation for support under various grants.

EP thanks the University of Guelph for a sabbatical leave in 2008–2009, during which the writing of this book was initiated. He is grateful to the Canadian Institute for Theoretical Astrophysics at the University of Toronto for its generous hospitality during this sabbatical. Research support from the Natural Sciences and Engineering Research Council is also gratefully acknowledged. The writing of this book coincided with a stint as department chair during the years 2008–2013; this project did much to preserve the sanity of the co-author.