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Calderón's inverse problem: imaging and invisibility

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We consider the determination of a conductivity function in a two-dimensional domain from the Cauchy data of the solutions of the conductivity equation on the boundary. In the first sections of the paper we consider this inverse problem, posed by Calderón, for conductivities that are in L^∞ and are bounded from below by a positive constant. After this we consider uniqueness results and counterexamples for conductivities that are degenerate, that is, not necessarily bounded from above or below. Elliptic equations with such coefficient functions are essential for physical models used in transformation optics and metamaterial constructions. The present counterexamples for the inverse problem have been related to invisibility cloaking. This means that there are conductivities for which a part of the domain is shielded from detection via boundary measurements. Such conductivities are called invisibility cloaks. At the end of the paper we consider the borderline of the smoothness required for the visible conductivities and the borderline of smoothness needed for invisibility cloaking conductivities.

1. Introduction

In electrical impedance tomography one aims to determine the internal structure of a body from electrical measurements on its surface. To consider the precise mathematical formulation of the electrical impedance tomography problem, suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with connected complement and let us start with the case when $\sigma : \Omega \to (0, \infty)$ be a measurable function that is bounded away from zero and infinity.

Then the Dirichlet problem

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \tag{1}$$

$$u\big|_{\partial\Omega} = \phi \in W^{1/2,2}(\partial\Omega) \tag{2}$$

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admits a unique solution u in the Sobolev space $W^{1,2}(\Omega)$. Here

$$W^{1/2,2}(\partial\Omega) = H^{1/2}(\partial\Omega) = W^{1,2}(\Omega)/W_0^{1,2}(\Omega)$$

stands for the space of equivalence classes of functions $W^{1,2}(\Omega)$ that are the same up to a function in $W_0^{1,2}(\Omega) = \mathrm{cl}_{W^{1,2}(\Omega)}(C_0^\infty(\Omega))$. This is the most general space of functions that can possibly arise as Dirichlet boundary values or traces of general $W^{1,2}(\Omega)$-functions in a bounded domain Ω.

In terms of physics, if the electric potential of a body Ω at point x is $u(x)$, having the boundary value $\phi = u|_{\partial\Omega}$, and there are no sources inside the body, u satisfies the equations (1)–(2). The electric current J in the body is equal to

$$J = -\sigma \nabla u.$$

In electrical impedance tomography, one measures only the normal component of the current, $\nu \cdot J|_{\partial\Omega} = -\nu \cdot \sigma \nabla u$, where ν is the unit outer normal to the boundary. For smooth σ this quantity is well defined pointwise, while for general bounded measurable σ we need to use the (equivalent) definition of $\nu \cdot \sigma \nabla u|_{\partial\Omega}$,

$$\langle \nu \cdot \sigma \nabla u, \psi \rangle = \int_\Omega \sigma(x) \nabla u(x) \cdot \nabla \psi(x)\, dm(x) \quad \text{for all } \psi \in W^{1,2}(\Omega)(\Omega), \tag{3}$$

as an element of $H^{-1/2}(\partial\Omega)$, the dual of space of $H^{1/2}(\partial\Omega) = W^{1/2,2}(\partial\Omega)$. Here, m is the Lebesgue measure.

Calderón's inverse problem is the question whether an unknown conductivity distribution inside a domain can be determined from the voltage and current measurements made on the boundary. The measurements correspond to the knowledge of the Dirichlet-to-Neumann map Λ_σ (or the equivalent quadratic form) associated to σ, i.e., the map taking the Dirichlet boundary values of the solution of the conductivity equation

$$\nabla \cdot \sigma(x) \nabla u(x) = 0 \tag{4}$$

to the corresponding Neumann boundary values,

$$\Lambda_\sigma : \ u|_{\partial\Omega} \mapsto \nu \cdot \sigma \nabla u|_{\partial\Omega}. \tag{5}$$

For sufficiently regular conductivities the Dirichlet-to-Neumann map Λ_σ can be considered as an operator from $W^{1/2,2}(\partial\Omega)$ to $W^{-1/2,2}(\partial\Omega)$. In the classical theory of the problem, the conductivity σ is bounded uniformly from above and below. The problem was originally proposed by Calderón [1980]. Sylvester and Uhlmann [1987] proved the unique identifiability of the conductivity in dimensions three and higher for isotropic conductivities which are C^∞-smooth, and Nachman [1988] gave a reconstruction method. In three dimensions or higher unique identifiability of the conductivity is proven for conductivities with $3/2$ derivatives [Päivärinta et al. 2003; Brown and Torres 2003] and $C^{1,\alpha}$-smooth conductivities which are C^∞ smooth outside surfaces on which they have conormal singularities [Greenleaf et al. 2003b]. Haberman and Tataru [2011] have recently proven uniqueness for the C^1-smooth conductivities. The problems has also been solved with measurements only on a part of the boundary [Kenig et al. 2007].

In two dimensions the first global solution of the inverse conductivity problem is due to Nachman [1996a] for conductivities with two derivatives. In this seminal paper the $\overline{\partial}$ technique was first time used in the study of Calderón's inverse problem. The smoothness requirements were reduced in [Brown and Uhlmann 1997a] to Lipschitz conductivities. Finally, in [Astala and Päivärinta 2006] the uniqueness of the inverse problem was proven in the form that the problem was originally formulated in [Calderón 1980], i.e., for general isotropic conductivities in L^∞ which are bounded from below and above by positive constants.

The Calderón problem with an anisotropic, i.e., matrix-valued, conductivity that is uniformly bounded from above and below has been studied in two dimensions [Sylvester 1990; Nachman 1996a; Lassas and Uhlmann 2001; Astala et al. 2005; Imanuvilov et al. 2010] and in dimensions $n \geq 3$ [Lee and Uhlmann 1989; Lassas and Uhlmann 2001; Ferreira et al. 2009]. For example, for the anisotropic inverse conductivity problem in the two-dimensional case it is known that the Dirichlet-to-Neumann map determines a regular conductivity tensor up to a diffeomorphism $F : \overline{\Omega} \to \overline{\Omega}$, i.e., one can obtain an image of the interior of Ω in deformed coordinates. This implies that the inverse problem is not uniquely solvable, but the nonuniqueness of the problem can be characterized. We note that the problem in higher dimensions is presently solved only in special cases, like when the conductivity is real analytic.

Electrical impedance tomography has a variety of different applications for instance in engineering and medical diagnostics. For a general expository presentations see [Borcea 2002; Cheney et al. 1999], for medical applications see [Dijkstra et al. 1993].

In the last section we will study the inverse conductivity problem in the two-dimensional case with degenerate conductivities. Such conductivities appear in physical models where the medium varies continuously from a perfect conductor to a perfect insulator. As an example, we may consider a case where the conductivity goes to zero or to infinity near ∂D where $D \subset \Omega$ is a smooth open set. We ask what kind of degeneracy prevents solving the inverse problem, that is, we study what is the border of visibility. We also ask what kind of degeneracy makes it even possible to coat of an arbitrary object so that it appears the same as a homogeneous body in all static measurements, that is, we study what is the border of the invisibility cloaking. Surprisingly, these borders are not the same; we identify these borderlines and show that between them there are the electric holograms, that is, the conductivities creating an illusion of a nonexisting body (see Figure 1 on page 43). These conductivities are the counterexamples for the unique solvability of inverse problems for which even the topology of the domain can not be determined using boundary measurements.

In this presentation we concentrate on solving Calderón's inverse problem in two dimensions. The presentation is based on the works [Astala and Päivärinta 2006; Astala et al. 2009; 2005; 2011a], where the problem is considered using quasiconformal techniques. In higher dimensions the usual method is to reduce, by substituting $v = \sigma^{1/2}u$, the conductivity equation (1) to the Schrödinger equation and then to apply the methods of scattering theory. Indeed, after such a substitution v satisfies

$$\Delta v - qv = 0,$$

where $q = \sigma^{-1/2}\Delta\sigma^{1/2}$. This substitution is possible only if σ has some smoothness. In the case $\sigma \in L^\infty$, relevant for practical applications the reduction to the Schrödinger equation fails. In the two-dimensional case we can overcome this by using methods of complex analysis. However, what we adopt from the scattering theory type approaches is the use of exponentially growing solutions, the so-called geometric optics solutions to the conductivity equation (1). These are specified by the condition

$$u(z,\xi) = e^{i\xi z}\left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right) \quad \text{as } |z| \to \infty, \tag{6}$$

where $\xi, z \in \mathbb{C}$ and ξz denotes the usual product of these complex numbers. Here we have set $\sigma \equiv 1$ outside Ω to get an equation defined globally. Studying the ξ-dependence of these solutions then gives rise to the basic concept of this presentation, the *nonlinear Fourier transform* $\tau_\sigma(\xi)$. The detailed definition will be given in Section 2F.

Thus to start the study of $\tau_\sigma(\xi)$ we need first to establish the existence of exponential solutions. Already here the quasiconformal techniques are essential. We note that the study of the inverse problems is closely related to the nonlinear Fourier transform: It is not difficult to show that the Dirichlet-to-Neumann boundary operator Λ_σ determines the nonlinear Fourier transforms $\tau_\sigma(\xi)$ for all $\xi \in \mathbb{C}$. Therefore the main difficulty, and our main strategy, is to invert the nonlinear Fourier transform, show that $\tau_\sigma(\xi)$ determines $\sigma(z)$ almost everywhere.

The properties of the nonlinear Fourier transform depend on the underlying differential equation. In one dimension the basic properties of the transform are fairly well understood, while deeper results such as analogs of Carleson's

L^2-convergence theorem remain open. The reader should consult the excellent lecture notes by Tao and Thiele [2003] for an introduction to the one-dimensional theory.

For (1) with nonsmooth σ, many basic questions concerning the nonlinear Fourier transform, even such as finding a right version of the Plancherel formula, remain open. What we are able to show is that for $\sigma^{\pm 1} \in L^\infty$, with $\sigma \equiv 1$ near ∞, we have a Riemann–Lebesgue type result,

$$\tau_\sigma \in C_0(\mathbb{C}).$$

Indeed, this requires the asymptotic estimates of the solutions (6), and these are the key point and main technical part of our argument. For results on related equations, see [Brown 2001]. The nonlinear Fourier transform in the two-dimensional case is also closely related to the Novikov–Veselov (NV) equation, which is a (2+1)-dimensional nonlinear evolution equation that generalizes the (1+1)-dimensional Korteweg-deVries(KdV) equation; see [Boiti et al. 1987; Lassas et al. 2007; Tsai 1993; Veselov and Novikov 1984].

2. Calderón's inverse for isotropic L^∞-conductivity

To avoid some of the technical complications, below we assume that the domain $\Omega = \mathbb{D} = \mathbb{D}(1)$, the unit disk. In fact the reduction of general Ω to this case is not difficult; see [Astala and Päivärinta 2006]. Our main aim in this section is to consider the following uniqueness result and its generalizations:

Theorem 2.1 [Astala and Päivärinta 2006]. *Let* $\sigma_j \in L^\infty(\mathbb{D})$, $j = 1, 2$. *Suppose that there is a constant* $c > 0$ *such that* $c^{-1} \le \sigma_j \le c$. *If*

$$\Lambda_{\sigma_1} = \Lambda_{\sigma_2},$$

then $\sigma_1 = \sigma_2$ *almost everywhere. Here* Λ_{σ_i}, $i = 1, 2$, *are defined by* (5).

For the first steps in numerical implementation of the solution of the inverse problem based on quasiconformal methods see [Astala et al. 2011b].

Our approach will be based on quasiconformal methods, which also enables the use of tools from complex analysis. These are not available in higher dimensions, at least to the same extent, and this is one of the reasons why the problem is still open for L^∞-coefficients in dimensions three and higher. The complex analytic connection comes as follows: From Theorem 2.3 below we see that if $u \in W^{1,2}(\mathbb{D})$ is a real-valued solution of (1), then it has the σ-harmonic conjugate $v \in W^{1,2}(\mathbb{D})$ such that

$$\partial_x v = -\sigma \partial_y u, \quad \partial_y v = \sigma \partial_x u. \tag{7}$$

Equivalently (see (26)), the function $f = u + iv$ satisfies the $\mathbb{R}$-linear Beltrami equation

$$\frac{\partial f}{\partial \bar z} = \mu(z)\overline{\frac{\partial f}{\partial z}}, \tag{8}$$

where $\frac{\partial f}{\partial \bar z} = \partial_{\bar z} f = \frac{1}{2}(\partial_x f + i\partial_y f)$, $\frac{\partial f}{\partial z} = \partial_{\bar z} f = \frac{1}{2}(\partial_x f - i\partial_y f)$, and

$$\mu = \frac{1-\sigma}{1+\sigma}.$$

In particular, note that μ is real-valued and that the assumptions on σ in Theorem 2.1 imply $\|\mu\|_{L^\infty} \le k < 1$. This reduction to the Beltrami equation and the complex analytic methods it provides will be the main tools in our analysis of the Dirichlet-to-Neumann map and the solutions to (1).

2A. ***Linear and nonlinear Beltrami equations.*** A powerful tool for finding the exponential growing solutions to the conductivity equation (including degenerate conductivities) are given by the nonlinear Beltrami equation. We therefore first review a few of the basic facts here. For more details and results see [Astala et al. 2009].

We start with general facts on the linear divergence-type equation

$$\operatorname{div} A(z)\nabla u = 0, \qquad z \in \Omega \subset \mathbb{R}^2 \tag{9}$$

where we assume that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and that the coefficient matrix

$$A = A(z) = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \qquad \alpha_{21} = \alpha_{12}, \tag{10}$$

is symmetric and elliptic:

$$\frac{1}{K(z)}|\xi|^2 \le \langle A(z)\xi, \xi\rangle \le K(z)|\xi|^2, \qquad \xi \in \mathbb{R}^2, \tag{11}$$

almost everywhere in Ω. Here, $\langle \eta, \xi\rangle = \eta_1\xi_1 + \eta_2\xi_2$ for $\eta, \xi \in \mathbb{R}^2$. We denote by $K_A(z)$ the smallest number for which (11) is valid. We start with the case when $A(z)$ is assumed to be isotropic, $A(z) = \sigma(z)\mathbf{I}$ with $\sigma(z) \in \mathbb{R}_+$. We also assume that there is $K \in \mathbb{R}_+$ such that $K_A(z) \le K$ almost everywhere.

For many purposes it is convenient to express the above ellipticity condition (11) in terms of the single inequality

$$|\xi|^2 + |A(z)\xi|^2 \le \left(K_A(z) + \frac{1}{K_A(z)}\right)\langle A(z)\xi, \xi\rangle \tag{12}$$

valid for almost every $z \in \Omega$ and all $\xi \in \mathbb{R}^2$. For the symmetric matrix $A(z)$ this is seen by representing the matrix as a diagonal matrix in the coordinates given by the eigenvectors.

Below we will study the divergence equation (9) by reducing it to the complex Beltrami system. For solutions to (9) a conjugate structure, similar to harmonic functions, is provided by the *Hodge star* operator $*$, which here really is nothing more than the (counterclockwise) rotation by 90 degrees,

$$* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad ** = -\mathbf{I}. \tag{13}$$

There are two vector fields associated with each solution to the homogeneous equation

$$\operatorname{div} A(z)\nabla u = 0, \qquad u \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

The first, $E = \nabla u$, has zero curl (in the sense of distributions, the curl of any gradient field is zero), while the second, $B = A(z)\nabla u$, is divergence-free as a solution to the equation.

It is the Hodge star $*$ operator that transforms curl-free fields into divergence-free fields, and vice versa. In particular, if

$$E = \nabla w = (w_x, w_y), \qquad w \in W^{1,1}_{\mathrm{loc}}(\Omega),$$

then $*E = (-w_y, w_x)$ and hence

$$\operatorname{div}(*E) = \operatorname{div}(*\nabla w) = 0,$$

at least in the distributional sense. We recall here a well-known fact from calculus (the Poincaré lemma):

Lemma 2.2. *Let $E \in L^p(\Omega, \mathbb{R}^2)$, $p \geq 1$, be a vector field defined on a simply connected domain Ω. If* $\operatorname{Curl} E = 0$, *then E is a gradient field; that is, there exists a real-valued function $u \in W^{1,p}(\Omega)$ such that $\nabla u = E$.*

When u is A-harmonic function in a simply connected domain Ω, that is, u solves the equation $\operatorname{div} A(z)\nabla u = 0$, then the field $*A\nabla u$ is curl-free and may be rewritten as

$$\nabla v = *A(z)\nabla u, \tag{14}$$

where $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is some Sobolev function unique up to an additive constant. This function v we call the *A-harmonic conjugate* of u. Sometimes in the literature one also finds the term *stream function* used for v.

The ellipticity conditions for A can be equivalently formulated for the induced complex function $f = u + iv$. We arrive, after a lengthy but quite routine purely algebraic manipulation, at the equivalent complex first-order equation for $f = u + iv$, which we record in the following theorem.

Theorem 2.3. *Let Ω be a simply connected domain and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a solution to*

$$\operatorname{div} A\, \nabla u = 0. \tag{15}$$

If $v \in W^{1,1}(\Omega)$ is a solution to the conjugate A-harmonic equation (14)*, the function $f = u + iv$ satisfies the homogeneous Beltrami equation*

$$\frac{\partial f}{\partial \bar z} - \mu(z)\frac{\partial f}{\partial z} - \nu(z)\,\overline{\frac{\partial f}{\partial z}} = 0. \tag{16}$$

The coefficients are given by

$$\mu = \frac{\alpha_{22} - \alpha_{11} - 2i\alpha_{12}}{1 + \operatorname{trace} A + \det A}, \qquad \nu = \frac{1 - \det A}{1 + \operatorname{trace} A + \det A}. \tag{17}$$

Conversely, if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{C})$ is a mapping satisfying (16)*, then $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$ satisfy* (14) *with A given by solving the complex equations in* (17)*:*

$$\alpha_{11}(z) = \frac{|1-\mu|^2 - |\nu|^2}{|1+\nu|^2 - |\mu|^2}, \tag{18}$$

$$\alpha_{22}(z) = \frac{|1+\mu|^2 - |\nu|^2}{|1+\nu|^2 - |\mu|^2}, \tag{19}$$

$$\alpha_{12}(z) = \alpha_{21}(z) = \frac{-2\operatorname{Im}(\mu)}{|1+\nu|^2 - |\mu|^2}, \tag{20}$$

The ellipticity of A can be explicitly measured in terms of μ and ν. The optimal ellipticity bound in (11) is

$$K_A(z) = \max\{\lambda_1(z), 1/\lambda_2(z)\}, \tag{21}$$

where $0 < \lambda_2(z) \le \lambda_1(z) < \infty$ are the eigenvalues of $A(z)$. With this choice we have pointwise

$$|\mu(z)| + |\nu(z)| = \frac{K_A(z) - 1}{K_A(z) + 1} < 1. \tag{22}$$

We also denote by $K_f(z)$ the smallest number for which the inequality

$$\|Df(z)\|^2 \le K_f(z) J(z, f) \tag{23}$$

is valid. Here, $Df(z) \in \mathbb{R}^2$ is the Jacobian matrix (or the derivative) of f at z and $J(z, f) = \det(Df(z))$ is the Jacobian determinant of f.

Below, let $k \in [0, 1]$ and $K \in [1, \infty]$ be constants satisfying

$$\sup_{z\in\Omega}(|\mu(z)| + |\nu(z)|) \le k \quad \text{and} \quad K := \frac{1+k}{1-k}. \tag{24}$$

Then (16) yields

$$\left|\frac{\partial f}{\partial \bar z}\right| \le k \left|\frac{\partial f}{\partial z}\right|.$$

The above ellipticity bounds have then the relation

$$K_f(z) \le K_A(z) \le K \quad \text{for a.e. } z \in \Omega. \tag{25}$$

A mapping $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfying (23) with $K_f(z) \le K < \infty$ is called a K-*quasiregular mappings*. If f is a homeomorphism, we call it K-*quasiconformal*. By Stoilow's factorization (Theorem A.9), any K-quasiregular mapping is a composition of holomorphic function and a K-quasiconformal mapping.

Remarks. 1. In this correspondence, ν is real valued if and only if the matrix A is symmetric.

2. A has determinant 1 if and only if $\nu = 0$ (this corresponds to the $\mathbb{C}$-linear Beltrami equation).

3. A is isotropic, that is, $A = \sigma(z)\mathbf{I}$ with $\sigma(z) \in \mathbb{R}_+$, if and only if $\mu(z) = 0$. For such A, the Beltrami equation (16) then takes the form

$$\frac{\partial f}{\partial \bar z} - \frac{1-\sigma}{1+\sigma}\,\overline{\frac{\partial f}{\partial z}} = 0. \tag{26}$$

2B. ***Existence and uniqueness for nonlinear Beltrami equations.*** Solutions to the Beltrami equation conformal near infinity are particularly useful in solving the equation.

When μ and ν as above have compact support and we have a $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ solution to the Beltrami equation $f_{\bar z} = \mu f_z + \nu \overline{f_z}$ in $\mathbb{C}$, where $f_{\bar z} = \partial_{\bar z} f$ and $f_z = \partial_z f$, normalized by the condition

$$f(z) = z + \mathbb{O}(1/z)$$

near ∞, we call f a *principal solution*. Indeed, with the Cauchy and Beurling transform (see the Appendix) we have the identities

$$\frac{\partial f}{\partial z} = 1 + \mathcal{S}\,\frac{\partial f}{\partial \bar z} \tag{27}$$

and

$$f(z) = z + \mathscr{C}\left(\frac{\partial f}{\partial \bar z}\right)(z), \qquad z \in \mathbb{C}. \tag{28}$$

Principal solutions are necessarily homeomorphisms. In fact we have the following fundamental *measurable Riemann mapping theorem:*

Theorem 2.4. *Let $\mu(z)$ be compactly supported measurable function defined in $\mathbb{C}$ with $\|\mu\|_{L^\infty} \le k < 1$. Then there is a unique principal solution to the Beltrami equation*

$$\frac{\partial f}{\partial \bar z} = \mu(z)\frac{\partial f}{\partial z} \quad \text{for almost every } z \in \mathbb{C},$$

and the solution $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$ is a K-quasiconformal homeomorphism of $\mathbb{C}$.

The result holds also for the general Beltrami equation with coefficients μ and ν; see Theorem 2.5 below.

In constructing the exponentially growing solutions to the divergence and Beltrami equations, the most powerful approach is by nonlinear Beltrami equations which we next discuss.

When one is looking for solutions to the general nonlinear elliptic systems

$$\frac{\partial f}{\partial \bar z} = H\Big(z, f, \frac{\partial f}{\partial z}\Big), \qquad z \in \mathbb{C},$$

there are necessarily some constraints to be placed on the function H that we now discuss. We write

$$H : \mathbb{C}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}.$$

We will not strive for full generality, but settle for the following special case. For the most general existence results, with very weak assumptions on H, see [Astala et al. 2009]. Here we assume

(1) the homogeneity condition, that $f_{\bar z} = 0$ whenever $f_z = 0$, equivalently,

$$H(z, w, 0) \equiv 0, \qquad \text{for almost every } (z, w) \in \mathbb{C}\times\mathbb{C};$$

(2) the uniform ellipticity condition, that for almost every $z, w \in \mathbb{C}$ and all $\zeta, \xi \in \mathbb{C}$,

$$|H(z, w, \zeta) - H(z, w, \xi)| \le k|\zeta - \xi|, \qquad 0 \le k < 1; \tag{29}$$

(3) the Lipschitz continuity in the function variable,

$$|H(z, w_1, \zeta) - H(z, w_2, \zeta)| \le C|\zeta|\,|w_1 - w_2|$$

for some absolute constant C independent of z and ζ.

Theorem 2.5. *Suppose $H : \mathbb{C}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ satisfies the conditions* (1)–(3) *above and is compactly supported in the z-variable. Then the uniformly elliptic nonlinear differential equation*

$$\frac{\partial f}{\partial \bar z} = H\Big(z, f, \frac{\partial f}{\partial z}\Big) \tag{30}$$

admits exactly one principal solution $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$.