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Leibnitz rules and the generalized Korteweg–de Vries equation

A primary role of this first chapter is motivational. We aim to convince the reader that *paraproducts* are important objects, which appear in analysis in a natural way. Paraproducts were discussed in Section 9.4 of Vol. I (i.e., Section I.9.4), in the context of the $T(1)$ theorem. Our goal here is in some sense complementary, since now we want to describe some of their connections to the theory of differential equations. We plan to do this in two steps. In the present chapter we explain the appearance of the Leibnitz rules, which play an important role in nonlinear PDEs and in Chapter 2 we show why paraproducts are the correct objects to use in understanding these estimates.

The Leibnitz rules are inequalities of the type

$$\|D^\alpha(fg)\|_r \lesssim \|D^\alpha f\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \|D^\alpha g\|_{q_2}{}^1. \tag{1.1}$$

which hold as long as $1 < p_i, q_i \leq \infty$, $1/r = 1/p_i + 1/q_i$ for $i = 1, 2$ and $1/(1+\alpha) < r < \infty$. The fractional derivative $D^\alpha h$ is defined for every $\alpha > 0$ by, as usual, $\widehat{D^\alpha h}(\xi) = (2\pi|\xi|)^\alpha \widehat{h}(\xi)$, and all the functions involved are defined on the real line. Such inequalities are valid in higher dimensions and also for an arbitrary number of functions, but for simplicity we restrict ourselves to this particular *bilinear* one-dimensional case. However, the method that we will develop to understand them works equally well in the general case.

There are many natural questions that the reader may have about these inequalities. Such questions will be addressed in detail in Chapter 2. For now, let us just point out that if instead of $D^\alpha(fg)$ one considers the simpler expressions fg or $(fg)'$ then the corresponding estimates follow easily from Leibnitz's formula and Hölder's inequality.

To motivate the inequalties (1.1), we shall describe some natural dispersive estimates for a certain generalized Korteweg–de Vries (gKdV) equation , which

[1] The sign $\lesssim$ means "less than or equal to within a multiplicative constant".

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rely (among other things) on such inequalities. However, there are other things that the reader will find in this chapter. In particular, in order to understand the so called *Airy function*, which will be defined in a natural way later on, we need to introduce *wave packets* and *phase-space portraits* concepts that will play a fundamental role throughout the rest of the book.

Let us start by considering the following initial-value problem (IVP) for the gKdV equation on the real line $\mathbb{R}$:

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x F(u) = 0, \\ u(0,x) = g(x), \end{cases} \tag{1.2}$$

where the solution $u(t,x)$ is a real-valued function of two real variables and the given function $g(x)$ is its initial profile. This equation models weakly nonlinear shallow-water waves. The above function F is continuous and satisfies $F(0) = 0$. In the particular case $F(x) = x^2/2$ we obtain the classical KdV equation. The following notations are standard.

For every time t, one naturally defines the function $u(t)$ by $u(t)(x) := u(t,x)$. Then, if B is an arbitrary Banach space, $C(\mathbb{R}, B)$ denotes the space of all B-valued continuous functions while $C_w(\mathbb{R}, B)$ denotes the space of all B-valued weakly continuous functions.

We will also rely on the following classical facts about the gKdV equation, which will be taken for granted hereafter.

First, if $F \in C^2$ and if $g \in H^1$ with $\|g\|_{H^1}$ small then the IVP (1.2) has a solution $u \in C(\mathbb{R}, L^2) \cap C_w(\mathbb{R}, H^1)$; here, we denote by H^1 the classical Sobolev space with one derivative in L^2. This is a classical theorem of Kato. Second, for $F(u) = |u|^s$ the gKdV equation has solitary-wave solutions of the form $u(t,x) = w(x - ct)$, called *solitons*. As one can see, these solutions do not change their shape and travel with speed c.

1.1. Conserved quantities

The interesting fact about the gKdV equation is that it has infinitely many conserved quantities (i.e., expressions involving the solution that remain constant in time). We list the first three of them:

(1) $\int_{\mathbb{R}} u\,dx$, *the conservation of mass;*
(2) $\int_{\mathbb{R}} u^2\,dx$, *conservation of the L^2 norm;*
(3) $\int_{\mathbb{R}} (\frac{1}{2}u_x^2 - V(u))\,dx$, *conservation of the Hamiltonian, where V is an integral of F.*

Here the function u is assumed to be smooth enough that these formulae are all well defined. Later, when we are going to use them, we will see that this will always be the case. The proofs below use the Fourier transform. The more standard approach based on integration by parts is left to the reader as one of the problems at the end of the chapter.

Proof of (1) Taking the Fourier transform (with respect to the x variable) of the equation, we obtain

$$\widehat{\partial_t u}(t,\xi) + (2\pi i\xi)^3 \widehat{u}(t,\xi) + (2\pi i\xi)\widehat{F(u)}(t,\xi) = 0, \tag{1.3}$$

from which we get $\widehat{\partial_t u}(t,0) = 0$. This implies that

$$\frac{d}{dt}\int_{\mathbb{R}} u(t,x)\,dx = 0$$

or, in other words, that $\int_{\mathbb{R}} u(t,x)\,dx$ is constant in time. □

Proof of (2) Here we assume for simplicity that $F(u) = u^5$. In fact, later on we will present dispersive estimates in this particular case.

In general, if f and g are real-valued functions then one has

$$\begin{aligned}\int_{\mathbb{R}} f(x)g(x)\,dx &= \int_{\mathbb{R}} \widehat{f}(\xi)\bar{\widehat{g}}(\xi)\,d\xi = \int_{\mathbb{R}} \widehat{f}(\xi)\widehat{g}(-\xi)\,d\xi \\ &= \int_{\mathbb{R}} \widehat{f}(\xi)\widehat{g}(-\xi)\,d\xi = \int_{\xi_1+\xi_2=0} \widehat{f}(\xi_1)\widehat{g}(\xi_2)\,d\xi_1 d\xi_2,\end{aligned}$$

where an overbar indicates the complex conjugate. In particular, we can write

$$\int_{\mathbb{R}} u^2\,dx = \int_{\xi_1+\xi_2=0} \widehat{u}(\xi_1)\widehat{u}(\xi_2)\,d\xi_1 d\xi_2$$

and so

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} u^2 dx \\ &\quad = 2\int_{\xi_2+\xi_2=0} \partial_t\widehat{u}(\xi_1)\,\widehat{u}(\xi_2)\,d\xi_1 d\xi_2.\end{aligned} \tag{1.4}$$

From (1.3) we know that

$$\partial_t\widehat{u}(\xi) = -(2\pi i\xi)^3\widehat{u}(\xi) - (2\pi i\xi)\widehat{F(u)}(\xi);$$

then (1.4) becomes

$$\begin{aligned}16\pi^3 i\int_{\xi_1+\xi_2=0} \xi_1^3\widehat{u}(\xi_1)\widehat{u}(\xi_2)\,d\xi_1 d\xi_2 - 4\pi i\int_{\xi_1+\xi_2=0} \xi_1\widehat{F(u)}(\xi_1)\widehat{u}(\xi_2)\,d\xi_1 d\xi_2 \\ = I + II.\end{aligned}$$

By symmetry, the first term, I, is equal to

$$8\pi^3 i \int_{\xi_1+\xi_2=0} \left(\xi_1^3+\xi_2^3\right) \widehat{u}(\xi_1)\widehat{u}(\xi_2)\, d\xi_1 d\xi_2,$$

which is clearly identically equal to zero. The second term, II, becomes

$$\begin{aligned}
&-4\pi i \int_{\xi_1+\xi_2=0} \xi_1 \widehat{u^5}(\xi_1)\widehat{u}(\xi_2)\, d\xi_1 d\xi_2 \\
&\quad = -4\pi i \int_{\xi_1+\xi_2=0} \xi_1 \left(\int_{\lambda_1+\cdots+\lambda_5=\xi_1} \widehat{u}(\lambda_1)\cdots\widehat{u}(\lambda_5) d\lambda_1\cdots d\lambda_5 \right) \widehat{u}(\xi_2)\, d\xi_1 d\xi_2 \\
&\quad = 4\pi i \int_{\lambda_1+\cdots+\lambda_5+\xi_2=0} \xi_2 \widehat{u}(\lambda_1)\cdots\widehat{u}(\lambda_5)\widehat{u}(\xi_2)\, d\lambda_1\cdots d\lambda_5 d\xi_2 \\
&\quad = \frac{4\pi}{6} i \int_{\lambda_1+\cdots+\lambda_5+\xi_2=0} (\lambda_1+\cdots+\lambda_5+\xi_2)\widehat{u}(\lambda_1)\cdots\widehat{u}(\lambda_5)\widehat{u}(\xi_2)\, d\lambda_1\cdots d\lambda_5 d\xi_2 \\
&\quad = 0.
\end{aligned}$$

This proves that $\int_{\mathbb{R}} u^2(t,x)\,dx$ indeed is independent of time. □

Proof of (3) As before, it is enough to show that the derivative with respect to time of the expression

$$\int_{\mathbb{R}} \left(\frac{1}{2} u_x^2 - V(u) \right) dx$$

is zero. Now one can write

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} \left(\frac{1}{2} u_x^2 - V(u) \right) dx \\
&\quad = \frac{d}{dt} \left(\frac{1}{2} \int_{\xi_1+\xi_2=0} \widehat{u}_x(\xi_1)\widehat{u}_x(\xi_2)\, d\xi_1 d\xi_2 - \int_{\mathbb{R}} V(u)\,dx \right) \\
&\quad = \frac{d}{dt} \left(-2\pi^2 \int_{\xi_1+\xi_2=0} \xi_1\xi_2 \widehat{u}(\xi_1)\widehat{u}(\xi_2)\, d\xi_1 d\xi_2 - \int_{\mathbb{R}} V(u)\,dx \right) = A + B.
\end{aligned}$$

By using the equality (1.3) we see that A equals

$$
\begin{aligned}
-4\pi^2 & \int_{\xi_1+\xi_2=0} \xi_1\xi_2\partial_t\widehat{u}(\xi_1)\widehat{u}(\xi_2)\,d\xi_1 d\xi_2 \\
&= -32\pi^5 i \int_{\xi_1+\xi_2=0} \xi_1\xi_2\xi_1^3\widehat{u}(\xi_1)\widehat{u}(\xi_2)\,d\xi_1 d\xi_2 \\
&\quad + 8\pi^3 i \int_{\xi_1+\xi_2=0} \xi_1\xi_2\xi_1\widehat{F(u)}(\xi_1)\widehat{u}(\xi_2)\,d\xi_1 d\xi_2 \\
&= 32\pi^5 i \int_{\xi_1+\xi_2=0} \xi_1^5\widehat{u}(\xi_1)\widehat{u}(\xi_2)\,d\xi_1 d\xi_2 \\
&\quad + 8\pi^3 i \int_{\xi_1+\xi_2=0} \xi_1^2\xi_2\widehat{F(u)}(\xi_1)\widehat{u}(\xi_2)\,d\xi_1 d\xi_2 \\
&= -8\pi^3 i \int_{\xi_1+\xi_2=0} \xi_1^3\widehat{F(u)}(\xi_1)\widehat{u}(\xi_2)\,d\xi_1 d\xi_2. \qquad (1.5)
\end{aligned}
$$

The last line follows since, as before, the first integral is zero by symmetry. Given that V is an integral of F, the second term B can be written as

$$
\begin{aligned}
-\int_{\mathbb{R}} F(u)\partial_t u\,dx &= -\int_{\xi_1+\xi_2=0} \widehat{F(u)}(\xi_1)\partial_t\widehat{u}(\xi_2)\,d\xi_1 d\xi_2 \\
&= -8\pi^3 i \int_{\xi_1+\xi_2=0} \widehat{F(u)}(\xi_1)\xi_2^3\widehat{u}(\xi_2)\,d\xi_1 d\xi_2 \\
&\quad + 2\pi i \int_{\xi_1+\xi_2=0} \xi_2\widehat{F(u)}(\xi_1)\widehat{F(u)}(\xi_2)\,d\xi_1 d\xi_2 \\
&= -8\pi^3 i \int_{\xi_1+\xi_2=0} \widehat{F(u)}(\xi_1)\xi_2^3\widehat{u}(\xi_2)\,d\xi_1 d\xi_2. \qquad (1.6)
\end{aligned}
$$

Now one observes that the sum of (1.5) and (1.6) is zero. □

1.2. Dispersive estimates for the linear equation

Let us consider now the *linear part* of the initial-value problem (IVP) (1.2), which is given by

$$
\begin{cases} \partial_t u + \partial_x^3 u = 0, \\ u(0) = g. \end{cases} \qquad (1.7)
$$

We will see that one can calculate the solution explicitly in this case. By taking the Fourier transform with respect to the x variable of the first equation, we obtain

$$\partial_t \widehat{u}(t,\xi) = 8\pi^3 i\xi^3 \widehat{u}(t,\xi),$$

while the second equation gives

$$\widehat{u}(0,\xi) = \widehat{g}(\xi).$$

By combining these two we obtain immediately

$$\widehat{u}(t,\xi) = \widehat{g}(\xi) e^{8\pi^3 it\xi^3}$$

or, equivalently, $u(t,x) = (g * K_t)(x) := S(t)g$, where

$$\widehat{K_t}(\xi) = e^{8\pi^3 it\xi^3}.$$

It is easy to observe that

$$K_t(x) = \frac{1}{(4\pi^2 t)^{1/3}} Ai\left(\frac{x}{(4\pi^2 t)^{1/3}}\right),$$

where Ai is the *Airy function* whose Fourier transform is $e^{2\pi i\xi^3}$. Of course, the functions K_t are defined a priori as distributions, but later we will see that they are in fact functions, whose asymptotic behavior will be studied in detail. The reader should also recall the related topics described in Chapter I.4.[2] The following lemma will play an important role.

Lemma 1.1 *One has the following:*
(i) *$Ai(x)$ is a bounded function and is $O(|x|^{-1/4})$ as $|x| \to \infty$;*
(ii) *$D^\alpha(Ai)$ is bounded for any $\alpha \in \left[0, \frac{1}{2}\right]$.*

The proof will be postponed to the end of the chapter. Using this lemma one can easily prove the following *dispersive estimates* for the solutions of the linear equation (1.7).

Lemma 1.2 *Let $g \in L^1$. Then*
(i) $\|S(t)g\|_\infty \lesssim t^{-1/3}\|g\|_1$.
(ii) $\|S(t)g\|_p \lesssim t^{(-1/3)(1-1/p)}\|g\|_1$ *for every* $p > 4$.

Proof To prove the first statement one can write

$$(S(t)g)(x) = (g * K_t)(x) = \int_{\mathbb{R}} K_t(x-y)g(y)\,dy,$$

[2] Chapter I.4 refers to Chapter 4 in Vol. I of the book.

from which one obtains

$$\begin{aligned}\|S(t)g\|_\infty &\lesssim \left\| \int_{\mathbb{R}} |K_t(\cdot - y)||g(y)|\,dy \right\|_\infty \\ &\lesssim t^{-1/3}\|g\|_1,\end{aligned}$$

using Lemma 1.1(i).

Similarly, to prove the second statement one can write

$$\|S(t)g\|_p \lesssim \int_{\mathbb{R}} \|K_t(\cdot - y)\|_p |g(y)|\,dy \leq \|K_t\|_p \|g\|_1,$$

and it is easy to see, again using Lemma 1.1(i) and the fact that $p > 4$, that

$$\|K_t\|_p \lesssim t^{(-1/3)(1-1/p)}. \qquad \square$$

The next three lemmas will also be useful later.

Lemma 1.3 *Let $g \in L^1$. Then*

$$\|D^{1/2}(S(t)g)\|_\infty \lesssim t^{-1/2}\|g\|_1.$$

Proof One has on the one hand

$$D^{1/2}\,(S(t)g) = D^{1/2}(K_t * g) = (D^{1/2}K_t) * g.$$

On the other hand,

$$\left|D^{1/2}K_t(x)\right| \lesssim \left| \frac{1}{t^{1/3}} (D^{1/2}Ai) \left(\frac{x}{(4\pi^2)t^{1/3}} \right) \frac{1}{t^{1/6}} \right| \lesssim \frac{1}{t^{1/2}},$$

using Lemma 1.1(ii). As a consequence,

$$\|D^{1/2}(S(t)g)\|_\infty \lesssim \|D^{1/2}K_t\|_\infty \|g\|_1 \lesssim t^{-1/2}\|g\|_1,$$

as desired. $\square$

Lemma 1.4 *Let $p \geq 2$ and $1/p + 1/p' = 1$. Then*

$$\left\|D^{1/2-1/p}(S(t)g)\right\|_p \lesssim t^{-1/2+1/p}\|g\|_{p'}.$$

Proof There are two distinct cases, which we have met already. For $p = 2$ the inequality follows immediately from Plancherel's theorem, while for $p = \infty$ it was the object of Lemma 1.3. The general case then follows from Stein's complex interpolation theorem, since one can clearly extend the definition of $D^\alpha h$ to complex exponents α. $\square$

Lemma 1.5 *Let $p \in \left(1, \frac{4}{3}\right)$. Then there exist $\beta > 0$, $\gamma > 0$ such that*

$$\left\| D^{\beta}(S(t)g) \right\|_{\infty} \lesssim t^{-\gamma} \left\| g \right\|_p \tag{1.8}$$

with $\beta \to \frac{1}{2}$ and $\gamma \to \frac{1}{2}$ as $p \to 1$.

Proof From Lemma 1.2 we know that the linear operator $S(t)$ maps L^1 into L^p for $p > 4$ with an operatorial bound of the type $O\left(t^{(-1/3)(1-1/p)}\right)$. As a consequence, by duality, since $S(t)$ is a convolution operator, it also maps $L^{p'}$ to L^{∞} with the same bound, which can be rewritten as $O\left(t^{-1/3p'}\right)$. Also, since $p > 4$ it follows that $1 < p' < \frac{4}{3}$. In other words, changing the notation a little, we have shown that

$$\|S(t)g\|_{\infty} \lesssim t^{-1/3p} \|g\|_p$$

for every $1 < p < \frac{4}{3}$. However, from Lemma 1.3 we know that

$$\left\| D^{1/2}(S(t)g) \right\|_{\infty} \lesssim t^{-1/2} \left\| g \right\|_1.$$

The desired conclusion (1.8) follows by complex interpolation between the two estimates above. □

It is also natural to ask what happens if one keeps the nonlinearity and instead drops the linear term in equation (1.2). In the case of the KdV equation (i.e. $F(x) = x^2/2$) the initial-value problem becomes

$$\begin{cases} \partial_t u + u \partial_x u = 0, \\ u(0, x) = g(x), \end{cases} \tag{1.9}$$

and it is a well-known fact that the solutions of (1.9) may develop *shocks*. For instance, one can check directly that for $g(x) = -x$ the solution is given by $u(t, x) = -x/(1 - t)$. While this is well defined for t strictly between 0 and 1, the solution breaks down and a shock is developed at time $t = 1$.

What we can conclude from this is that there is always a competition between the good influence of the linear term and the bad influence of the nonlinear term, in the IVP (1.2). It is remarkable that there is sometimes a perfect balance between the two, as one can see from the existence of solitons.

Exercise 1.1 Fill in the complex interpolation details in the proofs of Lemmas 1.4 and 1.5.

1.3. Dispersive estimates for the nonlinear equation

Given all these dispersive estimates that the solutions of the linear IVP (1.7) satisfy, it is natural to ask whether we have dispersion in the nonlinear case (1.2) as well. Clearly, assuming that $F(u) = |u|^s$, in general there is no dispersion because of the presence of solitons. However, it is natural to believe that if one starts with a small initial datum g, and if s is big enough, the dispersion should exist since then the nonlinear PDE is close to its linear counterpart.

More precisely, the goal of the section is to prove the following theorem.

Theorem 1.6 *Consider* (1.2) *with* $F(u) = u^5$. *Then there exists* $\varepsilon_0 > 0$ *such that if* g *satisfies*

$$\|g\|_1 + \|g\|_{H^1} < \varepsilon_0,$$

the solution to the corresponding gKdV equation is dispersive; more precisely, it satisfies

$$\sup_{t\in\mathbb{R}} \langle t\rangle^{1/3}\|u(t)\|_\infty < \infty. \tag{1.10}$$

In general by $\langle t\rangle$ one denotes the so-called *Japanese bracket* given by $\langle t\rangle = (1 + |t|^2)^{1/2}$. Let us also remark that it is crucial that the exponent of the nonlinearity be large enough ($s = 5$ in our case). For generic functions of the type $F(u) = |u|^s$, such a theorem does not hold for $1 < s < 3$, for example. To see this, let us recall the existence of solitons in this particular case. If

$$(t, x) \mapsto w(x - t)$$

is such a solution then a straightforward calculation shows that

$$(t, x) \mapsto \lambda^{1/(s-1)} w\left(\lambda^{1/2}x - \lambda^{3/2}t\right) \tag{1.11}$$

is also a solution, for every $\lambda > 0$. At time $t = 0$ this solution becomes $x \mapsto \lambda^{1/(s-1)} w\big(\lambda^{1/2}x\big) = w_\lambda(x)$. Then one has

$$\begin{aligned}
\|w_\lambda\|_1 + \|w_\lambda\|_{H^1} &\simeq \|w_\lambda\|_1 + \|w_\lambda\|_2 + \|w_\lambda'\|_2 \\
&= \lambda^{1/(s-1)-1/2}\|w\|_1 + \lambda^{1/(s-1)-1/4}\|w\|_2 + \lambda^{1/(s-1)+1/4}\|w'\|_2 \\
&= \lambda^{(3-s)/(2(s-1))}\|w\|_1 + \lambda^{(5-s)/(4(s-1))}\|w\|_2 \\
&\quad + \lambda^{(s+3)/(4(s-1))}\|w'\|_2.
\end{aligned}$$

Thus, if $1 < s < 3$ and λ is small enough, one can make $\|w_\lambda\|_1 + \|w_\lambda\|_{H^1}$ smaller than ε_0 while it is clear that the solution (1.11) is not dispersive.

To prove Theorem 1.6, we use a method of Christ and Weinstein. This method can be extended to cover more general nonlinearities, such as $F(u) = |u|^s$ for $s > 4$. However, since our goal here is mostly motivational we will describe it in the case $F(u) = u^5$, when many of its technicalities become easier. In fact, this particular case is part of an earlier result of Ponce and Vega.

Let us start by stating the following lemmas.

Lemma 1.7 *If g and u are as in Theorem 1.6 then, for any time t, one has*

$$\|u(t)\|_2 + \|\partial_x u(t)\|_2 \lesssim \|g\|_{H^1}.$$

Proof Let us remark that the inequality is completely trivial in the linear case. In the nonlinear case one observes that $\|u(t)\|_2 = \|u(0)\|_2 = \|g\|_2 \le \|g\|_{H^1}$, by the conservation of energy. It is therefore enough to prove the corresponding estimates for the term $\|\partial_x u(t)\|_2$.

We will first show that

$$\|\partial_x u(t)\|_2 \lesssim 1$$

for every t.

From the conservation of the Hamiltonian, we know that

$$\int_{\mathbb{R}} \left(\tfrac{1}{2}u_x^2 - V(u)\right) dx = \text{constant}$$

which implies that

$$\int_{\mathbb{R}} \left(u_x^2 - 2V(u)\right) dx = \text{constant}.$$

Then

$$\begin{aligned}
\|\partial_x u(t)\|_2^2 &= \int_{\mathbb{R}} u_x^2(t)\, dx = \int_{\mathbb{R}} \left(u_x^2(t) - 2V(u)(t)\right) dx + 2\int_{\mathbb{R}} V(u)(t)\, dx \\
&= \int_{\mathbb{R}} \left(u_x^2(0) - 2V(u)(0)\right) dx + 2\int_{\mathbb{R}} V(u)(t)\, dx \\
&= \int_{\mathbb{R}} u_x^2(0)\, dx + 2\left(\int_{\mathbb{R}} (V(u)(t) - V(u)(0))\, dx\right) \\
&\le \varepsilon_0^2 + 2\left|\int_{\mathbb{R}} V(u)(0)\, dx\right| + 2\left|\int_{\mathbb{R}} V(u)(t)\, dx\right|.
\end{aligned}$$

Also, since $V(u) = \frac{1}{6}u^6$ in our case, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}} V(u)(0)\, dx\right| &\lesssim \int_{\mathbb{R}} |u^6(0)|\, dx \le \|u(0)\|_\infty^4\, \|u(0)\|_2^2 \\
&\lesssim \|u(0)\|_{H^1}^4\, \|u(0)\|_{H^1}^2 = \|u(0)\|_{H^1}^6 \lesssim \varepsilon_0^6.
\end{aligned}$$