Analytic Combinatorics in Several Variables

Mathematicians have found it useful to enumerate all sorts of things arising in discrete mathematics: elements of finite groups, configurations of ones and zeros, graphs of various sorts; the list is endless. Analytic combinatorics uses analytic techniques to do the counting: generating functions are defined and their coefficients are then estimated via complex contour integrals. This book is the result of nearly fifteen years of work on developing analytic machinery to recover, as effectively as possible, asymptotics of the coefficients of a multivariate generating function. It is the first book to describe many of the results and techniques necessary to estimate coefficients of generating functions in more than one variable.

Aimed at graduate students and researchers in enumerative combinatorics, the book contains all the necessary background, including a review of the uses of generating functions in combinatorial enumeration as well as chapters devoted to saddle point analysis, Groebner bases, Laurent series and amoebas, and a smattering of differential and algebraic topology. All software along with other ancillary material can be located via the book website, www.cs.auckland.ac.nz/∼mcw/Research/mvGF/asymultseq/ACSVbook/.

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To the memory of Philippe Flajolet, on whose shoulders stands all of the work herein.

Contents

Preface

The term "analytic combinatorics" refers to the use of complex analytic methods to solve problems in combinatorial enumeration. Its chief objects of study are generating functions (Flajolet and Sedgewick, 2009, page vii). Generating functions have been used for enumeration for more than a hundred years, going back to Hardy and, arguably, to Euler. Their systematic study began in the 1950s (Hayman, 1956). Much of the impetus for analytic combinatorics comes from the theory of algorithms, arising, for example, in the work of Knuth (2006). The recent, seminal work by Flajolet and Sedgewick (2009) describes the rich univariate theory with literally hundreds of applications.

The multivariate theory, as recently as the mid-1990s, was still in its infancy. Techniques for *deriving* multivariate generating functions have been well understood, sometimes paralleling the univariate theory and sometimes achieving surprising depth (Fayolle, Iasnogorodski, and Malyshev, 1999). Analytic methods for recovering coefficients of generating functions once the functions have been derived have, however, been sorely lacking. A small body of analytic work goes back to the early 1980s (Bender and Richmond, 1983); however, even by 1995, of 100+ pages in the *Handbook of Combinatorics* devoted to asymptotic enumeration (Odlyzko, 1995), multivariate asymptotics received fewer than six.

This book is the result of work spanning nearly fifteen years. Our aim has been to develop analytic machinery to recover, as effectively as possible, asymptotics of the coefficients of a multivariate generating function. Both authors feel drawn to this area of study because it combines many areas of modern mathematics. Functions of one or more complex variables are essential, but also algebraic topology in the Russian style, stratified Morse theory, computational algebraic methods, saddle-point integration, and of course the basics of combinatorial enumeration. The many applications of this work in areas such as bioinformatics, queueing theory, and statistical mechanics are not surprising when we realize how widespread is the use of generating functions in applied combinatorics and probability.

The purpose of this book is to pass on what we have learned, so that others may learn it and use it before we forget it. The present form of the book grew out of graduate-level mathematics courses that developed, along with the theory, at the University of Wisconsin, Ohio State University, and the University of Pennsylvania. The course was intended to be accessible to students in their second year of graduate study. Because of the eclectic nature of the required background, this presents something of a challenge. One may count on students having seen calculus on manifolds by the end of a year of graduate studies, in addition to some complex variable theory. One may also assume some willingness to do some outside reading. However, some of the more specialized areas on which multivariate analytic combinatorics must draw are not easy to get from books. This includes topics such as the theory of amoebas (Gel'fand, Kapranov, and Zelevinsky, 1994) and the Leray-Petrovsky-Gårding theory of inverse Fourier transforms. Other topics such as saddle-point integration and stratified Morse theory exist in books but require being summarized to avoid a semester-long detour.

We have dealt with these problems by summarizing a great amount of background material. Part I contains the combinatorial background and will be known to students who have taken a graduate-level course in combinatorial enumeration. Part II contains mathematical background from outside of combinatorics. The topics in Part II are central to the understanding and execution of the techniques of analytic combinatorics in several variables. Part III contains the theory, all of which is new since the turn of the millennium and only parts of which exist in published form. Finally, there are appendices, almost equal in total size to Part II, which include necessary results from algebraic and differential topology. Some students will have seen these, but for the rest, the inclusion of these topics will make the present book self-contained rather than one that can only be read in a library.

We hope to recruit further researchers into this field, which still has many interesting challenges to offer, and this explains the rather comprehensive nature of the book. However, we are aware that some readers will be more focused on applications and seek the solution of a given problem. The book is structured so that after reading Chapter 1, it should be possible to skip to Part III and pick up supporting material as required from previous chapters. A list of publications using the multivariate methods described in this book can be found on our website: `www.cs.auckland.ac.nz/~mcw/Research/mvGF/asymultseq/ACSVbook/`.

The mathematical development of the theory belongs mostly to the two authors, but there are a number of individuals whose help was greatly instrumental in moving the theory forward. The complex analysts at the University of Wisconsin-Madison, Steve Wainger, Jean-Pierre Rosay, and Andreas Seeger, helped the authors (then rather junior researchers) to grapple with the problem in its earliest incarnation. A similar role was played several years later by Jeff McNeal. Perhaps the greatest thanks are due to Yuliy Baryshnikov, who translated the Leray-Petrovsky theory and the work of Atiyah-Bott-Gårding into terms the authors could understand and coauthored several articles. Frank Sottile provided help with algebra on many occasions; Persi Diaconis arranged for a graduate course while the

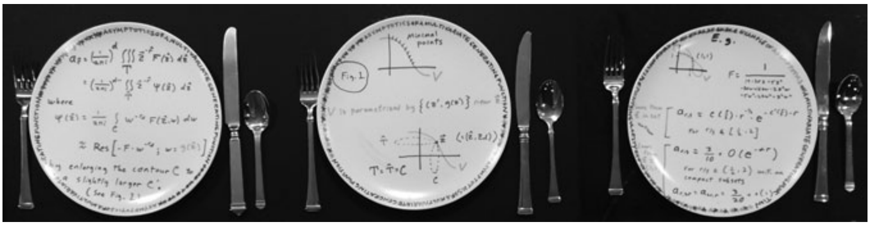

Figure 0.1 Customized "asymptotics of a multivariable generating function" dinner plates.

first author visited Stanford in 2000; Richard Stanley answered our numerous miscellaneous queries. Thanks are also due to our other coauthors on articles related to this project, listed on the project website linked from the book website. Alex Raichev and Torin Greenwood helped substantially with proofreading and with computer algebra implementations of some parts of the book. Thanks also to valuable proofreading contributions from Lily Yen. All software can be located via the book website.

On a more personal level, the first author would like to thank his wife, Diana Mutz, for encouraging him to follow this unusual project wherever it took him, even if it meant abandoning a still productive vein of problems in probability theory. The sentiment in the probability theory community may be otherwise, but the many connections of this work to other areas of mathematics have been a source of satisfaction to the authors. The first author would also like to thank his children, Walden, Maria, and Simi, for their participation in the project via the Make-A-Plate company (see Figure 0.1).

The second author thanks his wife Golbon Zakeri, children Yusef and Yahya, and mother-in-law Shahin Sabetghadam for their help in carving out time for him to work on this project, sometimes at substantial inconvenience to themselves. He hopes they will agree that the result is worth it.