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Part I

Spinless Bose and Fermi gases

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This part concerns non-relativistic models of spinless particles, localized on a 1D continuous line with periodic boundary conditions. We ignore the physical postulate which relates the spin and the statistics of identical particles and consider both spinless Bose and Fermi gases.

In chapter 1, we study integrable models with nearest-neighbor interactions. We introduce the most general version of singular point interactions. The interactions are defined either implicitly, through boundary conditions for the wavefunction at the singular point, or explicitly, in terms of the Dirac δ-function (δ-interactions) and its derivative (δ'-interactions). As a model system, we study the pure δ-interaction which is non-trivial only in the boson case. After a detailed derivation of the Bethe ansatz equations, the ground state of the model is studied. In the attractive interaction regime, the ground state in the sector of N particles corresponds to an N-string of complex momenta with energy $\propto N^3$ which leads to thermodynamic collapse. In the repulsive regime with all momenta real, the thermodynamic limit $N \to \infty$ of the ground state is well behaved. This is the opportunity to document Hultén's continualization procedure and to derive an integral equation for the ground-state particle density in momentum space which is analyzed in special limits of the interaction strength. An analogy with the disk condenser is pointed out. A system of particles with finite hard cores, impenetrable to other particles, is solved by the Bethe ansatz method as well.

The form of the Bethe ansatz equations is common for various integrable spinless Bose and Fermi gases; only a phase function depends on the particular type of pair interaction. The sequence of quantum numbers corresponding to the ground state is the same as well. This enables us to perform in chapter 2 a general analysis of low-lying excited states and of the zero-temperature thermodynamics. The analysis is based on the response of the system in the ground state to an external perturbation.

The derivation of the finite-temperature thermodynamics, based on standard thermodynamic principles combined with the hole concept of Yang, is the subject of chapter 3.

The 1D system of Bose and Fermi particles with inverse-square interaction is analyzed in chapter 4. In contrast to systems with contact interactions, the wavefunction becomes a superposition of plane waves only at asymptotically large distances between particles; this fact is behind the name "asymptotic Bethe ansatz". The model is obtained naturally as an answer to the following question: find a pair interaction for which the ground-state wavefunction is of pair-product form. Excited states, ground-state properties and finite-temperature thermodynamics of the model are also derived.

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1

Particles with nearest-neighbor interactions: Bethe ansatz and the ground state

1.1 General formalism

Elementary particles have internal degrees of freedom called spins (see Appendix A). A particle with spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ can be in one of $(2s+1)$ different states, i.e. the spin Hilbert space has dimension $(2s+1)$. The integer values of the spin $s = 0, 1, 2, \ldots$ correspond to bosons, the wavefunction of which is symmetric with respect to any interchange of two particles. The half-odd integer values $s = \frac{1}{2}, \frac{3}{2}, \ldots$ correspond to fermions, the wavefunction of which is antisymmetric with respect to interchange of two particles. The relation between the spin and the statistics of identical particles is a physical postulate. Since the boson and fermion systems are mathematically well defined for an arbitrary value of the spin, we shall ignore this physical postulate and consider both spinless Fermi and Bose particles.

Let a one-dimensional (1D) non-relativistic system consist of N identical spinless particles of mass m, denoted as $j = 1, 2, \ldots, N$. Their coordinates are constrained to a circle of circumference L, $0 \le x_j \le L$, with imposed periodic boundary conditions (BCs) $x + L \to x$. The thermodynamic limit $N, L \to \infty$ with the fixed particle density $n = N/L$ will be of special interest. The particles interact pairwisely by a reflection-symmetric potential $v(x-x') = v(x'-x)$ which is periodic, $v(x+L) = v(x)$. In units of $\hbar = 2m = 1$, the quantum many-body Hamiltonian reads

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j<k=1}^{N} v(x_j - x_k). \tag{1.1}$$

The energy spectrum of the system is determined by the time-independent Schrödinger equation

$$H\psi(x_1, x_2, \ldots, x_N) = E\psi(x_1, x_2, \ldots, x_N), \tag{1.2}$$

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where the wavefunction ψ satisfies periodic BCs

$$\psi(\ldots, x_j, \ldots) = \psi(\ldots, x_j + L, \ldots), \qquad j = 1, 2, \ldots, N. \tag{1.3}$$

The wavefunction ψ is determined up to a constant which is fixed by the normalization to unity of the integral over the N-particle coordinate space of $|\psi|^2$. The proper normalization of the wavefunction will be irrelevant and therefore ignored.

In the case of a single particle $N = 1$, the solution of the Helmholtz equation $-\psi''(x) = E\psi(x)$ is the plane wave with wavenumber (quasi-momentum) k:

$$\psi(x) = \frac{1}{\sqrt{L}} \exp(\mathrm{i}kx), \qquad E = k^2. \tag{1.4}$$

With regard to the periodic BC $\psi(x) = \psi(x + L)$, the wavenumber is quantized as $kL = 2\pi I$ ($I = 0, \pm 1, \pm 2, \ldots$). Since the particle system is formulated on a continuous line, the Hilbert space has infinite dimension and the possible values of the quantum number I are unbounded.

The spinless particles are supposed to obey boson/fermion statistics expressed by the symmetry/antisymmetry property of the wavefunction under the exchange of an arbitrary pair of particles. Since the system is 1D, the N particles can be ordered from left to right in $N!$ ways. Let S_N denote the symmetric group of all $N!$ permutations of the numbers $(12\ldots N)$. The ordering particle sectors will be labeled by the permutations $Q = (Q1, Q2, \ldots, QN) \in S_N$ according to the prescription $Q : \ 0 \le x_{Q1} < x_{Q2} < \cdots < x_{QN} \le L$. For example, in the case of $N = 3$ particles, $Q = (132)$ labels the ordering sector $x_1 < x_3 < x_2$, $Q = (213)$ labels the ordering sector $x_2 < x_1 < x_3$, etc. The *fundamental* sector, identified with the identity permutation $I = (12\ldots N)$, corresponds to the particle ordering $I : \ 0 \le x_1 < x_2 < \cdots < x_N \le L$. Each sector Q is adjacent to $N - 1$ other sectors at the points $x_{Q1} = x_{Q2}, x_{Q2} = x_{Q3}, \ldots, x_{Q(N-1)} = x_{QN}$.

Let us denote by $\psi_Q(x_1, x_2, \ldots, x_N)$ the wavefunction in the ordering sector Q and assume that $\psi_I(x_1, x_2, \ldots, x_N)$ is known. This function has no special exchange symmetry with respect to particle coordinates. In accordance with the standard symmetrization/antisymmetrization procedure, we have

$$\psi_Q(x_1, x_2, \ldots, x_N) = (\pm 1)^{\eta_Q} \psi_I(x_{Q1}, x_{Q2}, \ldots, x_{QN}), \tag{1.5}$$

where the $+/-$ sign corresponds to bosons/fermions and η_Q is the number of transpositions of nearest-neighbor elements which bring the permutation Q to I. We shall often use the notation $(-1)^{\eta_Q} = \mathrm{sign}(Q)$. The general wavefunction ψ with an arbitrary particle ordering can be formally represented as

$$\psi(x_1, x_2, \ldots, x_N) = \sum_{Q \in S_N} \theta_{\mathrm{H}}(x_{Q2} - x_{Q1}) \theta_{\mathrm{H}}(x_{Q3} - x_{Q2}) \cdots \theta_{\mathrm{H}}(x_{QN} - x_{Q(N-1)}) \psi_Q(x_1, x_2, \ldots, x_N), \tag{1.6}$$

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where $\theta_{\rm H}(x)$ is the Heaviside step function

$$\theta_{\rm H}(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases} \tag{1.7}$$

It is evident that $\theta'_{\rm H}(x) \equiv {\rm d}\theta_{\rm H}(x)/{\rm d}x = \delta(x)$, where $\delta(x)$ is the Dirac δ-function, and $\theta_{\rm H}(x) + \theta_{\rm H}(-x) = 1$.

To solve a many-body system, it is sufficient to find ψ_I. A minor problem is that the periodic BCs (1.3) relate different Q-sectors. Indeed, the shift of $x_1 = 0$ in the I-sector by L is identified with

$$\psi_I(x_1 = 0, x_2, \ldots, x_N) = \psi_Q(x_1 = L, x_2, \ldots, x_N), \qquad Q = (23 \ldots N1). \tag{1.8}$$

Here, since $x_1 = L$ is the largest of the particle coordinates, the wavefunction on the right-hand side corresponds to the ordering sector $Q = (23 \ldots N1)$ with $\eta_Q = N - 1$. The formula (1.5) allows us to obtain the periodic BC

$$\psi_I(x_1 = 0, x_2, \ldots, x_N) = (\pm 1)^{N-1} \psi_I(x_2, \ldots, x_N, x_1 = L) \tag{1.9}$$

which relates the wavefunctions in the same I-sector.

In the two-body problem, it is useful to pass to the center-of-mass inertia. Let us consider particle 1 with coordinate x_1 and particle 2 with coordinate x_2, moving on an infinite line $L \to \infty$. The time-independent Schrödinger equation for a two-body wavefunction $\psi(x_1, x_2)$ reads

$$-\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi + v(x_1 - x_2)\psi = E\psi. \tag{1.10}$$

The interaction potential v is assumed to be symmetric and goes to zero at asymptotically large distances between the particles,

$$v(x) = v(-x), \qquad \lim_{x \to \pm\infty} v(x) = 0. \tag{1.11}$$

When $|x_1 - x_2| \to \infty$, the free particles are described by plane waves with asymptotic momenta k_1, k_2 and energy $E = k_1^2 + k_2^2$. We define the total and relative asymptotic momenta as follows

$$K = k_1 + k_2, \qquad k = k_1 - k_2, \tag{1.12}$$

so the energy is $E = (K^2 + k^2)/2$. In the center-of-mass basis and relative coordinates

$$X = \frac{1}{2}(x_1 + x_2), \qquad x = x_1 - x_2, \tag{1.13}$$

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the wavefunction of the Schrödinger equation (1.10) takes a separation of variables form $\psi(x_1, x_2) = \mathrm{e}^{\mathrm{i}KX}\psi(x)$, where $\psi(x)$ satisfies the relative-motion equation

$$-\psi''(x) + \frac{1}{2}v(x)\psi(x) = \left(\frac{k}{2}\right)^2 \psi(x). \tag{1.14}$$

We see that the original two-body problem is reduced to the Schrödinger equation for one particle with energy $E = (k/2)^2$, moving in an "external" scattering potential $v(x)/2$.

For the symmetric potential (1.11), the differential equation (1.14) exhibits a solution of even (boson) parity

$$\psi_+(x) = \psi_+(-x), \qquad \psi'_+(x) = -\psi'_+(-x) \tag{1.15}$$

and a solution of odd (fermion) parity

$$\psi_-(x) = -\psi_-(-x), \qquad \psi'_-(x) = \psi'_-(-x). \tag{1.16}$$

1.2 Point interactions

It is believed that, in order to describe qualitatively low-energy properties of 1D systems with short-range interactions, one can replace these interactions by strictly local ones, singular and non-zero only at one point where two particles touch one another: $v(x) = 0$ for all $x \neq 0$ and $v(x) \to \pm\infty$ for $x = 0$. The first exactly solvable interaction of this type was the famous δ-function contact potential [21, 22]. The generalization to a larger class of singular interactions was accomplished by Šeba [75, 76]; for a pedagogical review, see [77].

We shall study the singular potentials within the one-body relative-motion equation (1.14). If the particle coordinate x is in the "free motion" domain $\Omega = \mathbb{R} - \{0\}$, the Hamiltonian consists of the kinetic energy operator $\mathbf{K} = \mathrm{d}^2/\mathrm{d}x^2$ only. The corresponding Hilbert space consists of the functions $\psi(x)$ with $x \in \Omega$ such that $\psi(x)$ and $\psi'(x)$ are absolutely convergent. There exist two equivalent representations of the singular potential, the implicit one based on the BC for the wavefunction at $x = 0$ and the explicit one in terms of the Dirac δ-function and its derivative.

The implicit representation comes from the fact that the singular potential $v(x)$ causes, in general, a discontinuity of $\psi(x)$ and $\psi'(x)$ at $x = 0$. Using the notation $f(0^{\pm}) \equiv \lim_{\varepsilon\to 0} f(\pm\varepsilon)$, the general BC can be expressed as

$$\begin{pmatrix} \psi(0^+) \\ \psi'(0^+) \end{pmatrix} = \mathbf{\Lambda} \begin{pmatrix} \psi(0^-) \\ \psi'(0^-) \end{pmatrix}, \qquad \mathbf{\Lambda} = \mathrm{e}^{\mathrm{i}\varphi} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.17}$$

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The requirement of the self-adjointness of the kinetic energy operator $\mathbf{K} = \mathbf{K}^\dagger$,

$$0 = \int_\Omega \mathrm{d}x \left[\psi^* \mathbf{K}\psi - (\mathbf{K}\psi)^*\psi\right] = -\int_\Omega \mathrm{d}x \left[\psi^*\psi'' - (\psi'')^*\psi\right]$$
$$= \left[\psi^*\psi' - (\psi')^*\psi\right](0^+) - \left[\psi^*\psi' - (\psi')^*\psi\right](0^-), \tag{1.18}$$

restricts the elements of the transfer matrix $\mathbf{\Lambda}$ to a four-parameter family

$$\varphi \in [0, \pi], \qquad a, b, c, d \in \mathbb{R}, \qquad ad - bc = 1. \tag{1.19}$$

Note that the self-adjointness condition (1.18) is equivalent to the requirement of the continuity of the probability current $j(x) = -\mathrm{i}[(\psi')^*\psi - \psi^*\psi']/2$ at $x = 0$. The phase parameter φ is redundant for stationary problems: it causes a constant phase shift between the wavefunctions $\psi(x > 0)$ and $\psi(x < 0)$, while the observables are independent of φ. The requirement of time-reversal invariance implies that $\exp(\mathrm{i}\varphi)$ should be real [78], say $\varphi = 0$.

The implicit formulation holds for both distinguishable and identical particles. The relationship between the general transfer matrix $\mathbf{\Lambda}$ (1.17) with parameters restricted by (1.19) and the scattering potential $v(x)$ is shown in Exercise 1.1.

The restriction to identical particles of boson parity (1.15) or fermion parity (1.16) leads to an additional equality $a = d$. Setting $a = d = 1$, there must hold $bc = 0$ and we obtain two one-parameter sets of transfer matrices

$$\mathbf{\Lambda}_\delta(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \qquad \mathbf{\Lambda}_{\delta'}(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \tag{1.20}$$

The first set $\mathbf{\Lambda}_\delta(c)$ preserves the continuity of the wavefunction while causing a jump of its derivative at $x = 0$. This corresponds to the δ-potential of strength c which has no effect on fermion (odd-parity) states due to the enforced constraint $\psi(0^+) = \psi(0^-) = 0$. The second set $\mathbf{\Lambda}_{\delta'}(b)$ preserves the continuity of the derivative while giving a jump in the wavefunction itself at $x = 0$. This contact interaction, known as the δ'-potential of strength b, has no effect on boson (even-parity) states due to the enforced constraint $\psi'(0^+) = \psi'(0^-) = 0$.

To be more specific, first we document the effect of the two transfer matrices (1.20) on the boson wavefunctions with even parity (1.15). In general, the boson wavefunction can be written as

$$\psi_+(x) = \theta_{\mathrm{H}}(x) f(x) + \theta_{\mathrm{H}}(-x) f(-x), \tag{1.21}$$

where $f(x)$ is a continuous function defined in the interval $x \in [0, \infty]$. It satisfies the Helmholtz-type equation $-f''(x) = (k/2)^2 f(x)$, whose general solution is

$$f(x) = \alpha \mathrm{e}^{\mathrm{i}kx/2} + \beta \mathrm{e}^{-\mathrm{i}kx/2}. \tag{1.22}$$

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The k-dependent parameters α and β are related to the BC for ψ_+ in the following way

$$\psi_+(0^+) = \psi_+(0^-) \equiv \psi_+(0) = f(0) = \alpha + \beta \tag{1.23}$$

and

$$\psi'_+(0^+) = -\psi'_+(0^-) = f'(0) = \frac{\mathrm{i}k}{2}(\alpha - \beta). \tag{1.24}$$

It is easy to show that the resulting boson wavefunction of the form

$$\psi_+(x) = \theta_{\mathrm{H}}(x)\left(\alpha \mathrm{e}^{\mathrm{i}kx/2} + \beta \mathrm{e}^{-\mathrm{i}kx/2}\right) + \theta_{\mathrm{H}}(-x)\left(\alpha \mathrm{e}^{-\mathrm{i}kx/2} + \beta \mathrm{e}^{\mathrm{i}kx/2}\right) \tag{1.25}$$

satisfies the relative-motion equation (1.14) with the potential

$$\frac{1}{2}v(x) = \mathrm{i}k\frac{\alpha - \beta}{\alpha + \beta}\delta(x). \tag{1.26}$$

Here, we applied the (symbolic) sifting property for the δ-function,

$$\delta(x)g(x) = g(0)\delta(x) \tag{1.27}$$

with $g(x)$ being a function continuous at $x = 0$, to the combination $\delta(x)\psi_+(0) = \delta(x)\psi_+(x)$. For the δ-transfer matrix $\mathbf{\Lambda}_\delta(c)$, we have

$$\frac{c}{2} = \frac{f'(0)}{f(0)} = \frac{\mathrm{i}k}{2}\frac{\alpha - \beta}{\alpha + \beta} \tag{1.28}$$

and (1.26) corresponds, in terms of the original particle coordinates, to the δ-interaction

$$v(x_1 - x_2) = 2c\delta(x_1 - x_2). \tag{1.29}$$

The BC (1.28) is written as

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi_+\Big|_{x_1 - x_2 \to 0^+} = c\psi_+\Big|_{x_1 = x_2}. \tag{1.30}$$

An equivalent BC follows from the interchange $x_1 \leftrightarrow x_2$. For the δ'-transfer matrix $\mathbf{\Lambda}_{\delta'}(b)$, we have $\alpha = \beta$ and the potential $v(x) = 0$ does not affect bosons.

The fermion wavefunction of the general form

$$\psi_-(x) = \theta_{\mathrm{H}}(x)f(x) - \theta_{\mathrm{H}}(-x)f(-x) \tag{1.31}$$

exhibits odd parity (1.16). The BCs for ψ_- now read

$$\psi_-(0^+) = -\psi_-(0^-) = f(0) = \alpha + \beta \tag{1.32}$$

and

$$\psi'_-(0^+) = \psi'_-(0^-) \equiv \psi'_-(0) = f'(0) = \frac{\mathrm{i}k}{2}(\alpha - \beta). \tag{1.33}$$

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The fermion wavefunction

$$\psi_-(x) = \theta_{\rm H}(x)\left(\alpha {\rm e}^{{\rm i}kx/2} + \beta {\rm e}^{-{\rm i}kx/2}\right) - \theta_{\rm H}(-x)\left(\alpha {\rm e}^{-{\rm i}kx/2} + \beta {\rm e}^{{\rm i}kx/2}\right) \tag{1.34}$$

satisfies the relative-motion equation (1.14) with the potential

$$\frac{1}{2}v(x) = \frac{4(\alpha+\beta)}{(\alpha-\beta){\rm i}k}\frac{\rm d}{{\rm d}x}\delta(x)\frac{\rm d}{{\rm d}x}. \tag{1.35}$$

Here, we applied the sifting relation (1.27) to the combination

$$\frac{\rm d}{{\rm d}x}\left[\delta(x)\psi_-'(0)\right] = \frac{\rm d}{{\rm d}x}\left[\delta(x)\frac{\rm d}{{\rm d}x}\right]\psi_-(x).$$

For the δ-transfer matrix $\mathbf{\Lambda}_\delta(c)$ in (1.20), we have $\alpha = -\beta$ and the potential $v(x) = 0$ does not affect fermions. For the δ'-transfer matrix $\mathbf{\Lambda}_{\delta'}(b)$, we have

$$\frac{2}{b} = \frac{f'(0)}{f(0)} = \frac{{\rm i}k}{2}\frac{\alpha-\beta}{\alpha+\beta}. \tag{1.36}$$

In view of (1.35), this corresponds to the δ'-interaction of type

$$v(x_1-x_2) = \frac{b}{2}\left(\frac{\rm d}{{\rm d}x_1} - \frac{\rm d}{{\rm d}x_2}\right)\delta(x_1-x_2)\left(\frac{\rm d}{{\rm d}x_1} - \frac{\rm d}{{\rm d}x_2}\right). \tag{1.37}$$

The BC (1.36) takes the form

$$\psi_-\Big|_{x_1-x_2\to 0^+} = \frac{b}{4}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi_-\Big|_{x_1=x_2}. \tag{1.38}$$

An equivalent BC is given by the interchange $x_1 \leftrightarrow x_2$.

It is interesting to compare the non-trivial cases of bosons with the δ-interaction (1.29) of coupling strength c and fermions with the δ'-interaction (1.37) of coupling strength b. From Eqs. (1.28) and (1.36) we see that if the two coupling constants are related by $cb = 4$, the building functions $f(x)$ of the two systems coincide. Thus a two-boson system with the δ-interaction is dual to a two-fermion system with the δ'-interaction, with the regions of strong and weak couplings reversed. The boson–fermion duality extends to arbitrary N-particle systems [79].

The singular interactions involving the δ-function and its derivative can be represented more physically as a superposition of interactions with a range tending toward zero [80, 81, 82].

As a model system we shall choose bosons with δ-interactions. With respect to the boson–fermion duality, its solution will automatically mean the exact solution of fermions with δ'-interactions.

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1.3 Bosons with δ-potential: Bethe ansatz equations

The Hamiltonian of the N-particle Bose system with δ-interactions reads

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<k} \delta(x_j - x_k). \tag{1.39}$$

The case $c = 0$ corresponds to free particles, $c < 0$ to particle attraction and $c > 0$ to particle repulsion. Particles can pass through each other, except for the impenetrable (point hard-core) limit $c = \infty$. When particle coordinates differ from each other, the δ-functions have no effect and the original Schrödinger equation can be replaced by a Helmholtz equation for free particles

$$-\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \psi_Q = E\psi_Q, \qquad 0 \le x_{Q1} < x_{Q2} < \cdots < x_{QN} \le L. \tag{1.40}$$

As was shown above, the presence of δ-functions in the Hamiltonian is equivalent to specific BCs of type (1.30) whenever two nearest-neighbor particles touch one another. For the symmetric Bose wavefunction in the fundamental ordering sector I, these conditions take the form

$$\left(\frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j}\right) \psi_I \Big|_{x_{j+1}-x_j \to 0^+} = c\psi_I \Big|_{x_{j+1}=x_j}, \qquad j = 1, \ldots, N-1. \tag{1.41}$$

Lieb and Liniger [21, 22] showed that the boson system is solvable by the coordinate Bethe ansatz. The usual way how to derive the Bethe ansatz equations is to increase successively the number of particles $N = 2, 3, \ldots$ in order to reveal the structure of the solution for an arbitrary N.

• $N = 2$: The wavefunction is sought as a superposition of plane waves with energy $E = k_1^2 + k_2^2$ in both particle ordering sectors $I \equiv (12)$ and (21):

$$\psi_I(x_1, x_2) = A(12)\mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2)} - A(21)\mathrm{e}^{\mathrm{i}(k_2x_1+k_1x_2)}, \tag{1.42}$$

$$\begin{aligned}\psi_{(21)}(x_1, x_2) &= +\psi_I(x_2, x_1) \\ &= A(12)\mathrm{e}^{\mathrm{i}(k_1x_2+k_2x_1)} - A(21)\mathrm{e}^{\mathrm{i}(k_2x_2+k_1x_1)},\end{aligned} \tag{1.43}$$

where the minus sign is attached to $A(21)$ for the sake of simplicity. The wavefunction is continuous at $x_1 = x_2$ for an arbitrary choice of the coefficients $A(12)$ and $A(21)$. The BC (1.41) taken at the contact point $x_1 = x_2$ implies the relation between the amplitudes $A(12)$ and $A(21)$:

$$\frac{A(21)}{A(12)} = \frac{k_2 - k_1 + \mathrm{i}c}{k_1 - k_2 + \mathrm{i}c} = \exp(-\mathrm{i}\theta_{12}), \tag{1.44}$$

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