Lyapunov Exponents

A Tool to Explore Complex Dynamics

Lyapunov exponents lie at the heart of chaos theory and are widely used in studies of complex dynamics. Utilising a pragmatic, physical approach, this self-contained book provides a comprehensive description of the concept. Beginning with the basic properties and numerical methods, it then guides readers through to the most recent advances in applications to complex systems. Practical algorithms are thoroughly reviewed and their performance is discussed, while a broad set of examples illustrates the wide range of potential applications. The description of various numerical and analytical techniques for the computation of Lyapunov exponents offers an extensive array of tools for the characterisation of phenomena, such as synchronisation, weak and global chaos in low- and high-dimensional setups, and localisation. This text equips readers with all of the investigative expertise needed to fully explore the dynamical properties of complex systems, making it ideal for both graduate students and experienced researchers.

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Preface

With the advent of electronic computers, numerical simulations of dynamical models have become an increasingly appreciated way to study complex and nonlinear systems. This has been accompanied by an evolution of theoretical tools and concepts: some of them, more suitable for a pure mathematical analysis, happened to be less practical for applications; other techniques proved instead very powerful in numerical studies, and their popularity exploded. Lyapunov exponents is a perfect example of a tool that has flourished in the modern computer era, despite having been introduced at the end of the nineteenth century.

The rigorous proof of the existence of well-defined Lyapunov exponents requires subtle assumptions that are often impossible to verify in realistic contexts (analogously to other properties, e.g., ergodicity). On the other hand, the numerical evaluation of the Lyapunov exponents happens to be a relatively simple task; therefore they are widely used in many setups. Moreover, on the basis of the Lyapunov exponent analysis, one can develop novel approaches to explore concepts such as hyperbolicity that previously appeared to be of purely mathematical nature.

In this book we attempt to give a panoramic view of the world of Lyapunov exponents, from their very definition and numerical methods to the details of applications to various complex systems and phenomena. We adopt a pragmatic, physical point of view, avoiding the fine mathematical details. Readers interested in more formal mathematical aspects are encouraged to consult publications such as the recent books by Barreira and Pesin (2007) and Viana (2014).

An important goal for us was to assess the reliability of numerical estimates and to enable a proper interpretation of the results. In particular, it is not advisable to underestimate the numerical difficulties and thereby use the various subroutines as black boxes; it is important to be aware of the existing limits, especially in the application to complex systems.

Although there are very few cases where the Lyapunov exponents can be exactly determined, methods to derive analytic approximate expressions are always welcome, as they help to predict the degree of stability, without the need of actually performing possibly long simulations. That is why, throughout the book, we discuss analytic approaches as well as heuristic methods based more on direct numerical evidence, rather than on rigorous theoretical arguments. We hope that these methods will be used not only for a better understanding of specific dynamical problems, but also as a starting point for the development of more rigorous arguments.

The various techniques and results described in the book started accumulating in the scientific literature during the 1980s. Here we have made an effort to present the main (according to our taste) achievements in a coherent and systematic way, so as to make the understanding by potentially unskilled readers easier. An example is the perturbative approach of the weak-disorder limit that has already been discussed in other reviews; here we present the case of ellyptic, hyperbolic and marginal matrices in a systematic manner.

Although this is a book and, as such, mostly devoted to a coherent presentation of known results, we have also included novel elements, wherever we felt that some gaps had to be filled. This is for instance, the case of the finite-size effects in the Kuramoto model or the extension of the techniques developed by Sompolinsky et al. to a wider class of random processes.

As a result, we are confident that the book can be read at various levels, depending on the needs of the reader. Those interested in the bare application to some simple cases will find the key elements in the first three chapters; the following chapters contain various degrees of in-depth analysis. Cross references among the common points addressed in the various sections should help the reader to navigate across specific items.

The most important acknowledgement goes to the von Humboldt Foundation, which, supporting the visit of Antonio Politi to Potsdam with a generous fellowship, has allowed us to start and eventually complete this project. Otherwise, writing the book would have been simply impossible.

We happened to discuss with, ask and receive suggestions from various colleagues. We specifically wish to acknowledge V. N. Biktashev, M. Cencini, H. Chaté, A. Crisanti, F. Ginelli, H. Kantz, R. Livi, Ya. Pesin, G. Puccioni, K. A. Takeuchi, R. Tonjes and H.-L. Yang.

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Special thanks go to P. Grassberger, who, more than 10 years after the publication of a joint paper with G. D'Alessandro, S. Isola and Antonio Politi on the Hénon map, was able to dig out some data to determine the still most accurate estimate of the topological entropy of such a map. As laziness has prevented a dissemination of those results, we made an effort to include them in this book.

We also wish to thank E. Lyapunova, the grand-niece of A. M. Lyapunov, who provided a high-quality photograph of the scientist who originated all of the story.

We finally warmly thank S. Capelin of Cambridge University Press, who has been patient enough to wait for us to complete the work. We hope that the delay has been worthy of a much better product. Although surely far from perfect, at some point we had to stop.