CAMBRIDGE
Cambridge University Press & Assessment
978-1-107-03025-1 — Cubical Homotopy Theory
Brian A. Munson , Ismar Volić
Frontmatter
More Information

Cubical Homotopy Theory

Graduate students and researchers alike will benefit from this treatment of various classical and modern topics in the homotopy theory of topological spaces, with an emphasis on cubical diagrams. The book contains more than 300 examples and provides detailed explanations of many fundamental results.

Part I focuses on foundational material on homotopy theory, viewed through the lens of cubical diagrams: fibrations and cofibrations, homotopy pullbacks and pushouts, and the Blakers–Massey Theorems. Part II includes a brief example-driven introduction to categories, limits, and colimits, and an accessible account of homotopy limits and colimits of diagrams of spaces. It also discusses cosimplicial spaces and relates this topic to the cubical theory of Part I, and provides computational tools via spectral sequences. The book finishes with applications to some exciting new topics that use cubical diagrams: an overview of two versions of calculus of functors and an account of recent developments in the study of the topology of spaces of knots.

BRIAN A. MUNSON is an Associate Professor of Mathematics at the US Naval Academy. He has held postdoctoral and visiting positions at Stanford University, Harvard University, and Wellesley College, Massachusetts. His research area is algebraic topology, and his work spans topics such as embedding theory, knot theory, and homotopy theory.

ISMAR VOLIĆ is an Associate Professor of Mathematics at Wellesley College, Massachusetts. He has held postdoctoral and visiting positions at the University of Virginia, Massachusetts Institute of Technology, and Louvain-la-Neuve University in Belgium. His research is in algebraic topology and his articles span a wide variety of subjects such as knot theory, homotopy theory, and category theory. He is an award-winning teacher whose research has been recognized by several grants from the National Science Foundation.

NEW MATHEMATICAL MONOGRAPHS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit www.cambridge.org/mathematics.

1. M. Cabanes and M. Enguehard *Representation Theory of Finite Reductive Groups*
2. J. B. Garnett and D. E. Marshall *Harmonic Measure*
3. P. Cohn *Free Ideal Rings and Localization in General Rings*
4. E. Bombieri and W. Gubler *Heights in Diophantine Geometry*
5. Y. J. Ionin and M. S. Shrikhande *Combinatorics of Symmetric Designs*
6. S. Berhanu, P. D. Cordaro and J. Hounie *An Introduction to Involutive Structures*
7. A. Shlapentokh *Hilbert's Tenth Problem*
8. G. Michler *Theory of Finite Simple Groups I*
9. A. Baker and G. Wüstholz *Logarithmic Forms and Diophantine Geometry*
10. P. Kronheimer and T. Mrowka *Monopoles and Three-Manifolds*
11. B. Bekka, P. de la Harpe and A. Valette *Kazhdan's Property (T)*
12. J. Neisendorfer *Algebraic Methods in Unstable Homotopy Theory*
13. M. Grandis *Directed Algebraic Topology*
14. G. Michler *Theory of Finite Simple Groups II*
15. R. Schertz *Complex Multiplication*
16. S. Bloch *Lectures on Algebraic Cycles (2nd Edition)*
17. B. Conrad, O. Gabber and G. Prasad *Pseudo-reductive Groups*
18. T. Downarowicz *Entropy in Dynamical Systems*
19. C. Simpson *Homotopy Theory of Higher Categories*
20. E. Fricain and J. Mashreghi *The Theory of H(b) Spaces I*
21. E. Fricain and J. Mashreghi *The Theory of H(b) Spaces II*
22. J. Goubault-Larrecq *Non-Hausdorff Topology and Domain Theory*
23. J. Śniatycki *Differential Geometry of Singular Spaces and Reduction of Symmetry*
24. E. Riehl *Categorical Homotopy Theory*
25. B. A. Munson and I. Volić *Cubical Homotopy Theory*
26. B. Conrad, O. Gabber and G. Prasad *Pseudo-reductive Groups (2nd Edition)*
27. J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson *Sobolev Spaces on Metric Measure Spaces*
28. Y.-G. Oh *Symplectic Topology and Floer Homology I*
29. Y.-G. Oh *Symplectic Topology and Floer Homology II*

Cubical Homotopy Theory

BRIAN A. MUNSON
United States Naval Academy, Maryland

ISMAR VOLIĆ
Wellesley College, Massachusetts

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107030251

First published 2015

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication data
Munson, Brian A., 1976–
Cubical homotopy theory / Brian A. Munson, United States Naval Academy, Maryland,
Ismar Volić, Wellesley College, Massachusetts.
pages cm
Includes bibliographical references and index.
ISBN 978-1-107-03025-1 (alk. paper)
1. Homotopy theory. 2. Cube. 3. Algebraic topology. I. Volić, Ismar, 1973–
II. Title.
QA612.7.M86 2015
514´.24–dc23
2015004584

ISBN 978-1-107-03025-1 Hardback

To Marissa and Catherine

Cambridge University Press & Assessment
978-1-107-03025-1 — Cubical Homotopy Theory
Brian A. Munson , Ismar Volić
Frontmatter
More Information

Contents

Preface

Purpose. Cubical diagrams have become increasingly important over the last two decades, both as a powerful organizational tool and because of their many applications. They provide the language necessary for the Blakers–Massey Theorems, which unify many classical results; they lie at the heart of calculus of functors, which has many uses in algebraic and geometric topology; and they are intimately related to homotopy (co)limits of diagrams and (co)simplicial spaces. The growing importance of cubical diagrams demands an up-to-date, comprehensive introduction to this subject.

In addition, self-contained, expository accounts of homotopy (co)limits and (co)simplicial spaces do not appear to exist in the literature. Most standard references on these subjects adopt the language of model categories, thereby usually sacrificing concreteness for generality. One of the goals of this book is to provide an introductory treatment to the theory of homotopy (co)limits in the category of topological spaces.

This book makes the case for adding the homotopy limit and colimit of a punctured square (homotopy pullback and homotopy pushout) to the essential toolkit for a homotopy theorist. These elementary constructions unify many basic concepts and endow the category of topological spaces with a sophisticated way to "add" (pushout) and "multiply" (pullback) spaces, and so "do algebra". Homotopy pullbacks and pushouts lie at the core of much of what we do and they build a foundation for the homotopy theory of cubical diagrams, which in turn provides a concrete introduction to the theory of general homotopy (co)limits and (co)simplicial spaces.

Features. We develop the homotopy theory of cubical diagrams in a gradual way, starting with squares and working up to cubes and beyond. Along the way, we show the reader how to develop competence with these topics with over 300 worked examples. Fully worked proofs are provided for the most part, and the reader will be able to fill in those that are not provided or have only been sketched. Many results in this book are known, but their proofs do not appear to exist. If we were not able to find a proof in the literature, we have indicated that this is the case. The reader will also benefit from an abundance of suggestions for further reading.

Cubical diagrams are an essential concept for stating and understanding the generalized Blakers–Massey Theorems, fundamental results lying at the intersection of stable and unstable homotopy theory. Our proofs of these theorems are new, purely homotopy-theoretic in nature, and use only elementary methods. We show how many important results, such as the Whitehead, Hurewicz, and Freudenthal Suspension Theorems, follow from the Blakers–Massey Theorems. Another new feature is our brief but up-to-date discussion of quasifibrations and of the Dold–Thom Theorem from the perspective of homotopy pullbacks and pushouts. Lastly, most of the material on spectral sequences of cubical diagrams also does not seem to have appeared elsewhere.

Our expositional preference is for (homotopy) limits rather than (homotopy) colimits. This is partly due to a quirk of the authors, but also because the applications in this book use (homotopy) limits more than (homotopy) colimits. We have, however, at least stated all the results in the dual way, but have omitted many proofs for statements about (homotopy) colimits that are duals of those for (homotopy) limits. If a proof involving (homotopy) colimits had something new to offer, we have included it.

Audience. A wide variety of audiences can benefit from reading this book. A novice algebraic topology student can learn the basics of some standard constructions such as (co)fibrations, homotopy pullbacks and pushouts, and the classical Blakers–Massey Theorem. A person who would like to begin to study the calculus of functors or its recent applications can read about cubical diagrams, the generalized Blakers–Massey Theorem, and briefly about calculus of functors itself. An advanced reader who does not want to adopt the cubical point of view can delve deeper into general homotopy (co)limits (while staying rooted in topological spaces and not going through the model-theoretic machinery that other literature adopts), cosimplicial spaces, or Bousfield–Kan spectral sequences. In addition, geometrically-minded topologists will appreciate some of main examples involving configuration spaces, applications to knot and link theory (and more general embedding spaces), as well as the geometric proof of the Blakers–Massey Theorem.

Organization. The book is naturally divided into two parts. The first two chapters of Part I can be thought of as the necessary background and may be skimmed or even skipped and returned to when necessary. The book really begins in Chapter 3, where we study squares, homotopy pullbacks and pushouts, and develop their arithmetic and algebraic properties. We can deduce many standard results in classical homotopy theory using this language, and this also provides the foundation of the theory for higher-dimensional cubes. The payoff at the end of this introductory material is in Chapter 4, where we present the Blakers–Massey Theorems for squares, tools for comparing homotopy pushouts and pullbacks. These are central results in homotopy theory, and we give many applications. We then move to higher-dimensional cubes in Chapter 5, and are able to bootstrap the material on squares to give an accessible account which generalizes most of the material encountered thus far. This treatment also provides a concrete introduction to more general homotopy (co)limits. Again we end with the Blakers–Massey Theorems in Chapter 6, but this time for higher-dimensional cubes.

Part II of the book explores more general categories and the definitions and main properties of homotopy (co)limits. The hope is that the reader will have acquired enough intuition through studying Part I, which is very concrete, to be able to transition to some of the general abstract notions of Part II. We review some general category theory in Chapter 7 but in Chapter 8 return to the category of topological spaces in order to preserve concreteness and continue to supply ample and familiar examples. We then move on to cosimplicial spaces in Chapter 9, which are closely allied both with the general theory of homotopy limits and with the cubical theory developed in Part I of the book. We end in Chapter 10 with a sequence of applications representing brief forays into current research, including introductory material of both homotopy and manifold calculus of functors and some applications. All of this uses the material developed earlier – cubical and cosimplicial machinery as well as the Blakers–Massey Theorem – in an essential way. An appendix serves to illustrate or give background on some topics which are used throughout the text but are not central to its theme, such as simplicial sets, spectra, operads, and transversality.

We have included a flowchart at the end of this preface to indicate the interdependence of the chapters.

Acknowledgments. Above all, we thank Tom Goodwillie, who taught us most of what we know about cubical diagrams. This project grew out of the notes from a series of lectures he gave us while we were still graduate students. His influence and generosity with the mathematics he has passed on to us cannot be overstated.

We thank Phil Hirschhorn for his generous help, technical and mathematical. His careful reading and input into various parts of the book, especially Chapters 2, 8, and 9, was invaluable.

We thank Greg Arone for many helpful conversations and for his encouragement and enthusiasm for this project.

We also thank Emanuele Dotto, Dan Dugger, John Oprea, Ben Walter, and the anonymous reviewers for useful comments and suggestions.

Lastly, we thank the editors and the staff at Cambridge University Press for their support and patience during the writing of this book.

BRIAN A. MUNSON
ISMAR VOLIĆ

Flowchart. The dotted lines in the chart below indicate that a reader who has had some exposure to category theory can skip Chapter 7, which contains fairly standard material (there might need to be an occasional look back at this chapter for notation or statements of results). The left and the right columns of the chart are precisely Parts I and II.