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PART I

Cubical diagrams

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Preliminaries

This chapter establishes some notational conventions and fundamental constructions. It is not comprehensive, nor is all of it even necessary (the unnecessary bits are meant to supply context), and we assume the reader is familiar with most of it already. Some of the material presented in this chapter is redundant in the sense that it will be revisited later. For instance, the cone, wedge, and suspension of a space will be discussed later in terms of colimits, a perspective more in line with the philosophy of this book. The essential topics presented here which are utilized elsewhere are topologies on spaces and spaces of maps, homotopy equivalences, weak equivalences, and a few properties of the class of CW complexes whose extra structure we will need from time to time. Some familiarity with homotopy groups (mostly their definition) will also be useful, and to a much lesser extent some exposure to homology. Many proofs are omitted, and references are given instead. We will clarify which is which along the way.

Most references given in this chapter are from Hatcher's *Algebraic topology* [Hat02]. There are a few other modern references which the authors have found useful, and which contain most, if not all, of these preliminary results as well, such as [AGP02, Gra75, May99, tD08] (we especially like [AGP02] since it seems to be the most elementary text which follows this book's philosophy; [Gra75] is neither modern nor in print, but still a unique and valuable resource). We owe all of these sources a debt, in this chapter and elsewhere.

1.1 Spaces and maps

A topological space is a pair (X, τ), where τ is a collection of subsets of X, the members of which are called *open* sets, which contains both the empty set and X, and which is closed under finite intersections and arbitrary unions. However, it is customary to suppress the topology from the notation, so we simply write X in place of (X, τ), and typically denote generic topological spaces using capital Roman letters. A *subbase* for a topology τ on X is a subset of τ for which every element of τ is a union of finite intersections of elements in the subset; it is a sort of generating set for τ.

The complement of an open set $U \subset X$ is called *closed*, and to specify a topology on X we may equivalently describe a system of closed sets which contains both the empty set and X, and which is closed under arbitrary intersections and finite unions. If $A \subset X$, we write $\mathring{A}$ for its *interior*; $\mathring{A}$ is the union of all open sets in X which are contained in A, and as such is an open set. We will write $\overline{A}$ for the *closure* of A; by definition it is the intersection of all the closed subsets of X which contain A, and is evidently closed.

By a *subspace* $A \subset X$, we mean the topology on A whose open sets are of the form $A \cap U$ where U is open in X. A collection of subspaces of a space X is called a *cover* of X if their union is X. For a nested sequence $X_1 \subset X_2 \subset \cdots$ of topological spaces, we endow the union $X = \bigcup_{i=1}^{\infty} X_i$ with the *weak topology*: a subset $C \subset X$ is closed if $C \cap X_i$ is closed in X_i for each i. For a topological space X with an equivalence relation $\sim$, we let $X/\sim$ denote the set of equivalence classes of X under $\sim$, and endow this space with the *quotient topology*: a set of equivalence classes is called open if the union of those equivalence classes forms an open set in X. For x in X, we denote by $[x]$ the corresponding point in $X/\sim$. For the quotient of X by a subspace A, we write X/A for this space and mean the quotient of X by the equivalence relation $\sim$ on X generated by $x \sim y$ if $x, y \in A$.

We will write $X \times Y$ for the product, $X \sqcup Y$ for the disjoint union, and when X and Y are subspaces of some larger space, $X \cup Y$ for the union along the intersection (which may be empty). Open sets of the product are generated by products of open sets, and open sets in the disjoint union are disjoint unions of open sets.

Several spaces are worthy of mention.

Definition 1.1.1

- The empty set $\emptyset$, that is, the space with no points.
- The one-point space $*$.
- The real numbers $\mathbb{R}$, topologized using the metric $d(x, y) = |x - y|$.
- The unit interval $I = \{t \in \mathbb{R} : 0 \leq t \leq 1\}$, topologized as a subspace of $\mathbb{R}$.
- Euclidean n-space $\mathbb{R}^n$, topologized using the usual metric $| - |$. By definition $\mathbb{R}^0 = \{0\}$.

- The n-dimensional cube $I^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : 0 \leq t_i \leq 1 \text{ for all } i\}$, topologized as a subspace of $\mathbb{R}^n$.
- The unit n-disk $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ for $n \geq 0$, topologized as a subset of $\mathbb{R}^n$. We define $D^{-1} = \emptyset$.
- The unit $(n-1)$-sphere $S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n : |x| = 1\}$ for $n \geq 0$, topologized as a subspace of $\mathbb{R}^n$. Note that $S^{-1} = \emptyset$; we define $S^{-2} = \emptyset$.
- The n-simplex $\Delta^n = \{(t_1, \ldots, t_n) : 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}$. Alternatively, $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : 0 \leq x_i \leq 1 \text{ for all } i \text{ and } \sum_i x_i = 1\}$ (it is a standard exercise to prove the two descriptions are equivalent; when we have need of coordinates in the simplex we will utilize the latter description). A simplex is topologized as a subspace of the Euclidean space of which it is a subset. For $0 \leq k \leq n$, let $\partial_k \Delta^n \subset \Delta^n$ denote the subset of Δ^n consisting of those tuples $(x_0, \ldots, x_n)$ for which $x_k = 0$. This is called a *face* of Δ^n, and it is itself a simplex of dimension $n - 1$.

We assume our topological spaces to be compactly generated Hausdorff. To be Hausdorff means that any two distinct points are contained in disjoint open neighborhoods.

Definition 1.1.2 A space X is said to be *compactly generated* if it has the property that a subset C of X is closed if and only if the intersection $C \cap K$ with each compact subset K of X is also closed in X.

All of the spaces described in Definition 1.1.1 are compactly generated Hausdorff. We will denote by Top the category of compactly generated spaces. The definition of a category can be found in Definition 7.1.1 (and we will not use the language of categories in a serious way before Chapter 7). Any Hausdorff space X can be made into a compactly generated space kX (same point set, different topology: we take the smallest compactly generated topology which contains the given one). The identity function $kX \to X$ is continuous and a homeomorphism if and only if X is compactly generated. Moreover, kX and X have the same compact subsets and the same homotopy groups (defined below). The product of two compactly generated spaces is given the topology of $k(X \times Y)$. Locally compact Hausdorff spaces, manifolds, metric spaces, and CW complexes (see below) are all compactly generated spaces. One benefit of working with compactly generated spaces is that this makes the duality between the notions of cofibration and fibration cleaner to state by eliminating the hypothesis of local compactness.

Maps between spaces will typically be denoted by a lower-case Latin letter such as f or g; thus $f : X \to Y$ denotes a map between the topological spaces

X and Y. In this case we say X is the *domain* of f and Y the *codomain*. The term "map" means continuous map.

Several maps are worthy of mention.

Definition 1.1.3

- The *identity map* from a space X to itself will be denoted by 1_X, defined by $1_X(x) = x$ for all x.
- The map from X to Y which has constant value $y \in Y$ will be denoted $c_y\colon X \to Y$, and is referred to as a *constant map*.
- For a subspace $A \subset X$, we usually write $\iota\colon A \to X$ for the inclusion map; occasionally we may use i for this map. The subspace A is a *retract* of X if there exists a map $r\colon X \to A$, called a *retraction*, such that $r \circ \iota = 1_A$.
- Given a map $f\colon X \to Y$, we say a map $g\colon Y \to X$ is a *section* of f if the composite $f \circ g = 1_Y$. Thus the inclusion map for a subspace which is a retract is a section of the retraction.
- For a space X with equivalence relation $\sim$, there is a *quotient map* $q\colon X \to X/\sim$ which sends each point to its equivalence class.
- For a space X, we write $\Delta\colon X \to X \times X$ for the *diagonal map*, defined by $\Delta(x) = (x, x)$.
- For a space X, we write $\nabla\colon X \coprod X \to X$ for the *fold map*, defined to be the identity 1_X on each summand.
- Given a map $f\colon X \to Y$ and a subspace $A \subset X$, we let $f|_A\colon A \to Y$ denote the *restriction* of f to A.
- A map $f\colon X \to Y$ is called a *homeomorphism* if there exists a continuous inverse $g\colon Y \to X$ for f; i.e. we have $g \circ f = 1_X$ and $f \circ g = 1_Y$. Spaces X and Y are then *homeomorphic* and we write $X \cong Y$. This is the most basic equivalence relation on the class of spaces we will consider.

Here is a useful result we will need later.

Lemma 1.1.4 *If X is a Hausdorff space and A is a retract of X, then A is closed in X.*

Proof Let $r\colon X \to A$ be the retraction and consider the map $X \to X \times X$ given by $x \mapsto (x, r(x))$. The preimage of the diagonal in $X \times X$ is the set of fixed points of r, which is A. But since X is Hausdorff, the diagonal is closed, and hence A is closed. □

Returning again to a nested sequence $X_1 \subset X_2 \subset \cdots$ of topological spaces, we note that the weak topology on $X = \bigcup_{i=1}^{\infty} X_i$ has the property that, given a collection of maps $f_i\colon X_i \to Y$ such that $f_i|_{X_{i-1}} = f_{i-1}$ for all i, there is a unique map $f\colon X \to Y$ whose restriction to X_i is equal to f_i for all i.

Often we consider *diagrams* of spaces, which simply means families of spaces and maps between them. The language is chosen to suggest we are thinking of a sort of picture of these spaces and maps. We will usually deal with *commutative* diagrams, which means that all ways of getting from one space to another by following maps are the same. The two we will most frequently encounter are

$$\begin{array}{ccc} X & \xrightarrow{\quad h \quad} & Y \\ & {\scriptstyle f}\searrow \qquad \nearrow{\scriptstyle g} & \\ & Z & \end{array}$$

where commutativity means $g \circ f = h$ and

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

where commutativity means $g \circ f' = f \circ g'$. We typically omit drawing the composed map $W \to Z$ in squares such as the above. We will later use the language of categories and functors to talk about diagrams (see Remark 7.1.16). If a diagram is not necessarily commutative, we will be explicit about this. One notable generally non-commutative diagram we will encounter first in Chapter 7 is

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

where $f \neq g$.

The spaces in a diagram will often be parametrized by subsets of some finite set, and so we shall encounter spaces such as X_U to denote which member of the family of spaces labeled "X" we mean. In the case that U is a subset of a finite set, say $U = \{1, 2\} \subset \{1, 2, 3\}$, we will usually write X_{12} in place of $X_{\{1,2\}}$ for cleaner presentation.

We will also often consider *pairs* of spaces (X, A). where X is a space and $A \subset X$ is a subspace. A *map of pairs* $f\colon (X, A) \to (Y, B)$ is a map $f\colon X \to Y$ such that $f(A) \subset B$. When $A = \{x_0\}$ is a single point, then X will be called *based* or *pointed*, x_0 will be called the *basepoint*, and we will typically write (X, x_0) in place of $(X, \{x_0\})$. A map $f\colon X \to Y$ of based spaces X and Y with basepoints x_0 and y_0 respectively is *based* if it is a map of pairs (X, x_0) and (Y, y_0). We let Top_* denote the category of based spaces (meaning the objects of interest are based spaces and the maps of interest are based maps). A map of pairs $f\colon (X, A) \to (Y, B)$ is called a *homeomorphism of pairs* if it has a continuous inverse $g\colon (Y, B) \to (X, A)$. We can also consider based pairs (X, A, x_0), where $x_0 \in A$ is the basepoint, and consider based maps $(X, A, x_0) \to (Y, B, y_0)$ of pairs, whose definition should be apparent.

The basepoint will often be suppressed from notation, but the reader should be warned that many constructions which require a choice of basepoint are not independent of that choice, such as the fundamental group. This should not cause confusion, as we will be clear about choosing basepoints when we are forced to do so.

Definition 1.1.5 A based space (X, x_0) is called *well-pointed* if $X \times \{0\} \cup \{x_0\} \times I$ is a retract of $X \times I$. If (X, x_0) is well-pointed, we call the basepoint x_0 *non-degenerate*. We assume all spaces to be well-pointed.

Remark 1.1.6 We prefer the equivalent definition that the inclusion of the basepoint $\{x_0\} \to X$ is a cofibration, but we do not have this language available yet. We will revisit this definition in Remark 2.3.20 once we have established the notion of a cofibration. The reason we assume our spaces to be well-pointed is to preserve homotopy invariance of various standard constructions, such as the suspension. □

CW complexes, mentioned above, play an important role at various points in this text and so it is worth recalling at least the idea of their construction. For instance, in Chapter 8 we will frequently deal with the realization of a simplicial complex, and it is easy to see how these can be considered as CW complexes. We refer the reader to [Hat02, Chapter 0, Appendix A] for more details on CW complexes. An *n-cell* e^n is simply the n-disk D^n, and $\partial e^n = \partial D^n$ is its boundary, the $(n-1)$-sphere. A CW complex X is a space built inductively starting with the empty set $X^{-1} = \emptyset$, with X^n built from X^{n-1} by attaching cells e^n_α to X^{n-1} via maps $a_\alpha\colon \partial e^n \to X^{n-1}$. Here α ranges through a (possibly empty) indexing set A_n. Thus X^0 is a discrete set of points, and in general X^n is a quotient space of $X^{n-1} \sqcup_{\alpha \in A_n} e^n_\alpha$. The space X is then defined as $\cup_{n \geq 0} X^n$ and is given the weak topology: A subset $C \subset X$ is closed if $C \cap X^n$ is closed in X^n for all n. We call X^n the *n-skeleton* of X. A *subcomplex* of a CW complex X is a subset A which is a union of cells of X such that the closure of each cell is contained in A.

A *relative CW complex* is a pair (X, A) where A is a topological space and X a space which has been built from A by attaching cells as above. That is, we use the same definition as above only with $X^{-1} = A$. The case where

$A = \emptyset$ specializes to an ordinary CW complex, so we may assume all CW complexes are relative CW complexes for the purposes of any statements about such spaces. The space A need not have a cell structure itself. A relative CW complex (X, A) has *dimension* n if $X = X^n$ and $X \neq X^{n-1}$. We say it is *finite* if the number of its cells is finite; that is, each indexing set A_n above is finite and there exists N such that $A_n = \emptyset$ for $n \geq N$. Theorem 1.3.7 below says that any space can be approximated by a CW complex in a suitable sense.

The notion of CW complex furnishes an enormous number of examples of topological spaces built from disks. We now review some others basic constructions: the cone, suspension, join, wedge, and smash product of spaces. We will encounter all of these again in Chapters 2 and 3 as examples of homotopy (co)fibers and homotopy (co)limits, and will derive many results combining and comparing them there.

Definition 1.1.7 For a topological space X, define X_+ to be X with a disjoint basepoint.

The space X_+ is in fact the quotient of X by the empty set. This is easiest to see diagrammatically by consideration of universal properties, as in Example 3.5.6.

Definition 1.1.8 For a space X, the *cone* CX on X is the quotient space

$$CX = X \times I/(X \times \{1\}).$$

If $X = \emptyset$, then CX is a point. If X is based with basepoint x_0, the *reduced cone* on X, by abuse also called CX, is the quotient space

$$X \times I/(X \times \{1\} \cup \{x_0\} \times I).$$

Any map $f: X \to Y$ induces a map of cones,

$$Cf: CX \to CY, \tag{1.1.1}$$

induced from the map $f \times 1_I : X \times I \to Y \times I$ by taking quotient spaces of the domain and codomain. See Example 3.6.8 for an alternative definition of the cone.

Remark 1.1.9 The quotient map from the cone to the reduced cone is a homotopy equivalence because (X, x_0) is well-pointed by assumption. See Remark 2.3.20 for a discussion. □

The cone on X is so named as it is created from the cylinder $X \times I$ by collapsing one end. The space X is naturally a subspace of CX by the inclusion of $X \times \{0\}$ in CX.

Definition 1.1.10 For a based space X with basepoint x_0, the *reduced suspension* ΣX of X is the quotient space

$$\Sigma X = X \times I/(X \times \{0\} \cup \{x_0\} \times I \cup X \times \{1\}).$$

Its basepoint is the image of $X \times \{0\} \cup \{x_0\} \times I \cup X \times \{1\}$ by the quotient map. The *unreduced suspension* of a space X, based or not, is the quotient space

$$X \times I/\sim$$

where $\sim$ is the equivalence relation generated by $(x_1, 0) \sim (x_2, 0)$ and $(x_1, 1) \sim (x_2, 1)$. Inductively we define $\Sigma^n X = \Sigma\Sigma^{n-1}X$.

The suspension is the union of two copies of the reduced cone CX glued together by the identity map $X \times \{0\} \to X \times \{0\}$. See Example 3.6.9 for an alternative definition of suspension.[1]

Remark 1.1.11 We will also use ΣX to denote the unreduced suspension, and we will refer to both versions simply as the *suspension*. It should always be clear to the reader which version we mean, usually depending on the existence of a basepoint, although we will try to be clear in instances where this may cause confusion. In any case, if X is well-pointed (see Remark 1.1.6), then the quotient map from the unreduced suspension to the reduced suspension of X is a homotopy equivalence, so this is usually not a concern. □

Example 1.1.12 It is an easy exercise to see that there is a homeomorphism $\Sigma S^n \cong S^{n+1}$ for all $n \geq -1$. (The sphere S^{-1} is empty, but its suspension is the quotient of the empty set by two points, and hence may be identified with S^0.) □

Any map $f: X \to Y$ induces a map of suspensions,

$$\Sigma f : \Sigma X \to \Sigma Y, \tag{1.1.2}$$

induced from the map $f \times 1_I : X \times I \to Y \times I$.

[1] Using the language of Section 2.4, another way to think of this is as the cofiber of the cofibration $X \to CX$.