Repeated Games

Three leading experts have produced a landmark work based on a set of working papers published by the Center for Operations Research and Econometrics (CORE) at Université Catholique de Louvain in 1994, under the title "Repeated Games," which holds almost mythic status among game theorists. Jean-François Mertens, Sylvain Sorin, and Shmuel Zamir have significantly elevated the clarity and depth of presentation with many results presented at a level of generality that goes far beyond the original papers – many written by the authors themselves. Numerous results are new, and many classic results and examples are not to be found elsewhere. Most remain state of the art in the literature. This book is full of challenging and important problems that are set up as exercises, with detailed hints provided for their solution. A new bibliography traces the development of the core concepts up to the present day.

The late Jean-François Mertens (11 March 1946–17 July 2012) was professor at the Université Catholique de Louvain (where he earned his PhD) and a member of the CORE. One of the world's leading experts in game theory and economic theory, Mertens is the author of seminal papers on equilibrium selection in games, formulation of Bayesian analysis, repeated and stochastic games, general equilibrium, social choice theory, and dynamic general equilibrium. A Fellow of the Econometric Society, he was also a founding member of the Center for Game Theory in Economics at the State University of New York at Stony Brook.

Sylvain Sorin is a member of the Mathematics Department at the Université Pierre et Marie Curie. He was previously professor at Université L. Pasteur, Strasbourg, and Université Paris X-Nanterre. He has been an affiliated member of the Département de Mathématiques, École Normale Supérieure (Paris); the Laboratoire d'Econométrie, École Polytechnique, Palaiseau; and the Center for Game Theory in Economics, State University of New York at Stony Brook. He is a Fellow of the Econometric Society and was a charter member of the Game Theory Society and editor-in-chief of the *International Journal of Game Theory*.

Shmuel Zamir, author of the textbook *Game Theory* (with M. Maschler and E. Solan, Cambridge 2013), is professor emeritus at the Hebrew University of Jerusalem and a founding member of the Center for the Study of Rationality there. He is professor of economics at the University of Exeter Business School, UK. Zamir is a Fellow of the Econometric Society, a charter member and a former council member of the Game Theory Society, and an affiliated member of the Center for Game Theory in Economics at the State University of New York at Stony Brook. Since 2008, he has been the editor-in-chief of the *International Journal of Game Theory*.

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Repeated Games

Jean-François Mertens

Sylvain Sorin
Université Pierre et Marie Curie, France

Shmuel Zamir
The Hebrew University of Jerusalem, Israel, and the University of Exeter, UK

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Foreword

Robert J. Aumann

John von Neumann reportedly said that pure and applied mathematics have a symbiotic relationship: not only does applied math draw heavily on the tools developed on the pure side, but, correspondingly, pure math cannot exist in the rarefied atmosphere of abstract thought alone; if it is not somehow rooted in the real world, it will wither and die.

The work before us – which certainly qualifies as beautiful, subtle, pure mathematics – is a case in point. It originated half a century ago, at the height of the Cold War between the United States and the Soviet Union, indeed as a direct result of that conflict. The US and SU were trying to keep the Cold War from getting hot; to minimize the damage if it did; and to cut down the enormous expenses that the nuclear arms race entailed. To that end, they met repeatedly in Geneva to negotiate mutual reductions in their nuclear arsenals. Regarding these arsenals, both sides were in the dark. Neither knew how many weapons the other had; and clearly, it was the number retained, rather than destroyed, that mattered. In Princeton, Oskar Morgenstern and Harold Kuhn had just founded the mathematics consulting firm "Mathematica." The United States Arms Control and Disarmament Agency (ACDA) was responsible for conducting the Geneva negotiations for the US; it turned to Mathematica to see whether the Theory of Games – created two decades earlier by John von Neumann and Oskar Morgenstern (the same as the Mathematica principal) – could help in addressing the strategic issues raised by these negotiations. Mathematica responded by assembling a team of theorists that included Gerard Debreu, John Harsanyi, Harold Kuhn, Mike Maschler, Jim Mayberry, Herb Scarf, Reinhard Selten, Martin Shubik, Dick Stearns, and the writer of these lines. Mike and I took charge of the informational aspect (Dick joined us later): whether one side could glean any information about the size of the other's nuclear arsenal from its tactics in previous negotiation rounds. To get a handle on this problem, we started by looking at the simplest possible analogues: very simple-looking two-person zero-sum repeated games, in which one player knows the payoff matrix while the other does not, and each observes the action of the other at each stage of the repetition. In such games, can the uninformed player glean any information about the payoff matrix from the informed player's actions at previous stages? Answering this question, even for the simplest 2×2 games, turned out to be surprisingly difficult – and challenging, fun! I vividly remember feeling that we were not working on a contrived, artificial problem, but were exploring the mysteries of the real world, like an astronomer or biologist. Thus was born the theory of repeated games of incomplete information.

What developed from that early work certainly *cannot* be considered applied math. To be sure, some insights may have been useful; for example, that in the context of a long series of repetitions, one cannot make use of information without implicitly revealing it. As a very practical corollary, we told the ACDA that it might be advisable to withhold some information from the ACDA's own negotiators. But the lion's share of the theory did not become directly useful, neither at that time nor subsequently. It really is pure mathematics: though *inspired* by experience – by the real world – it is of no direct use, at least to date.

The theory born in the mid to late sixties under the Mathematica-ACDA project started to grow and develop soon thereafter. For many years, I was a frequent visitor at CORE – the Center for Operations Research and Econometrics – founded in the late sixties by Jacques Drèze as a unit of the ancient university of Leuven-Louvain in Belgium. Probably my first visit was in 1968 or '69, at which time I met the brilliant, flamboyant young mathematician Jean-François Mertens (a little reminiscent of John Nash at MIT in the early fifties). One Friday afternoon, Jean-François took me in his Alfa-Romeo from Leuven to Brussels, driving at 215 km/hour, never slowing down, never sounding the horn, just blinking his lights – and indeed, the cars in front of him moved out of his way with alacrity. I told him about the formula, in terms of the concavification operator, for the value of an infinitely repeated two-person zero-sum game with one-sided incomplete information – which is the same as the limit of values of the n-times repeated games. He caught on immediately; the whole conversation, including the proof, took something like five or ten minutes. Those conversations – especially the vast array of fascinating, challenging open problems – hooked him; it was like taking a mountain climber to a peak in the foothills of a great mountain range, from where he could see all the beautiful unclimbed peaks. The area became a lifelong obsession with him; he reached the most challenging peaks.

At about the same time, Shmuel Zamir, a physics student at the Hebrew University, asked to do a math doctorate with me. Though a little skeptical, I was impressed by the young man, and decided to give it a try. I have never regretted that decision; Shmuel became a pillar of modern game theory, responsible for some of the most important results, not to speak of the tasks he has undertaken for the community. One problem treated in his thesis is estimating the error term in the above-mentioned limit of values; his seminal work in that area remains remarkable to this day. When Maschler and I published our Mathematica-ACDA reports in the early nineties, we included postscripts with notes on subsequent developments. The day that our typist came to the description of Zamir's work, a Jerusalem bus was bombed by a terrorist, resulting in many dead and wounded civilians. By a slip of the pen – no doubt Freudian – she typed "terror term" instead of "error term." Mike did not catch the slip, but I did, and to put the work in its historical context, purposely refrained from correcting it; it remains in the book to this day.

After finishing his doctorate, Shmuel – like many of my students – did a postdoctoral stint at CORE. While there, he naturally met up with Jean-François, and an immensely fruitful lifelong collaboration ensued. Together they attacked and solved many of the central unsolved problems of Repeated Game theory.

One of their beautiful results concerns the limit of values of n-times repeated two-person zero-sum games with incomplete information on *both* sides – like the original repeated Geneva negotiations, where neither the US nor the SU knew how many nuclear weapons the other side held. In the Mathematica-ACDA work, Maschler, Stearns, and I had shown that the infinite repetition of such games need not have a value: the minmax may be strictly greater than the maxmin. Very roughly, that is because, as mentioned above, using information involves revealing it. The minmax is attained when the maximizing player uses his information, thereby revealing it; but the minimizing player refrains from using her information until she has learned the maximizing player's information, and so can use *it*, in addition to her own. The maxmin is attained in the opposite situation, when he waits for her. In the infinitely repeated game, no initial segment affects the payoff, so each side waits for the other to use its information; the upshot is that there is *no* value – no way of playing a "long" repetition optimally, if you don't know *how* long it is.

But in the n-times repeated game, you can't afford waiting to use your information; the repetition will eventually end, rendering your information useless. Each side must use its information gradually, right from the start, thereby gradually revealing it; simultaneously, each side gradually learns the information revealed by the other, and so can – and does – use it. So it is natural to ask whether the values converge – whether one can speak of the value of a "long" repetition, without saying *how* long. Mike, Dick, and I did not succeed in answering this question. Mertens and Zamir did: they showed that the values indeed converge. Thus one *can* speak of the *value* of a "long" repetition without saying how long, even though one cannot speak of optimal *play* in such a setting. This result was published in the first issue – Vol. 1, No. 1 – of the *International Journal of Game Theory*, of which Zamir is now, over forty years later, the editor.

The Mertens–Zamir team made many other seminal contributions. Perhaps best known is their construction of the complete type space. This is not directly related to repeated games, but rather to all incomplete information situations – it fully justifies John Harsanyi's ingenious concept of "type" to represent multi-agent incomplete information.

I vividly remember my first meeting with Sylvain Sorin. It was after giving a seminar on repeated games (of complete information, to the best of my recall) in Paris, sometime in the late seventies, perhaps around 1978 or '79. There is a picture in my head of standing in front of a grand Paris building, built in the classical style with a row of Greek columns in front, and discussing repeated games with a lanky young French mathematician who actually understood everything I was saying – and more. I don't remember the contents of the conversation; but the picture is there, in my mind, vividly.

There followed years and decades of close cooperation between Sylvain, Jean-François, Shmuel, and other top Israeli mathematical game theorists. Sylvain and Jean-François came to Israel frequently, and the Israelis went to France and Belgium frequently. One winter, Sylvain and his family even joined me and my family for a few days of skiing in the Trois Vallées. During those years, Sylvain succeeded in attracting an amazing group of students, which became today's magnificent French school of mathematical game theory. One summer, he came to the annual game theory festival at Stony Brook University with twelve doctoral students; "Sylvain and his apostles" were the talk of the town.

Of the book's three authors, only Sylvain actually conducted joint research with the writer of these lines. We conjectured a result during the conference on repeated games organized by Abraham Neyman at the Israel Academy of Sciences in the spring of 1985; concentrated work on it started at the 1985–6 emphasis year in Math Econ and Computation organized by Gerard Debreu at the Mathematical Sciences Research Institute in Berkeley, in which we both participated; it continued by correspondence after we each returned to our home bases; finally, we succeeded in proving the conjecture, and in 1989 published it as the first paper in Vol.1, No.1, of the journal *Games and Economic Behavior*. The result concerns endogenous emergence of cooperation in a repeated game, and perhaps that is a good place to wrap up this preface. The book before us has been in the making, in one sense or another, for close to half a century; so its production may well be viewed as a repeated – or dynamic – game. And, both the production of the book itself, and the work described therein, have been highly cooperative ventures, spanning decades and continents.

The above has been a highly personal account of my involvement with the people and the work that made this extraordinary book happen. I have not done justice to the book itself. Perhaps the best way to do so is to quote from the reports of the anonymous readers who were asked by the publisher to report on the book. These reports are uniformly excellent and highly enthusiastic – I wish my work got reports like that. We here content ourselves with the opening paragraph of just one of those reports; the enthusiastic tone is typical:

> The results and proofs in this text are the foundations on which modern repeated-game theory is built. These are results that apply to zero-sum games, stochastic games, repeated games of incomplete information, spaces of beliefs, stochastic processes and many many other topics. It is impossible to find these results together in one place except in this volume. Existing texts and monographs cover some of them, but none covers anything like all of these topics. However, it is not the coverage of foundational material that makes this text one of a kind; it is the generality and the breadth of vision that is its most special feature. In virtually every section and result the authors strive

> to establish the most powerful and most general statement. The intellectual effort required to produce this work is huge. It was an enormous undertaking to have brought these results together in this one place. This makes the work as a whole sound leaden and dull; however, it is anything but that. It is filled with an intellectual *joie de vivre* that delights in the subject. This is epitomized by the astonishing links between disparate topics that are casually scattered throughout its pages – the Minmax Theorem used to prove the Peron–Frobenius Theorem; the Normal distribution arising in repeated games with incomplete information; the use of medial limits as a way of describing payoffs. . . .

It should be added that the book provides encyclopedic coverage of the area of repeated games – with and without complete information – as well as of stochastic and other dynamic games. The main emphasis is on developments during the classical period – the second half of the twentieth century – during which the theory took shape. Later developments – right up to the present – are also thoroughly covered, albeit more briefly.

In short, the work before us is an extraordinary intellectual tour de force; I congratulate and salute the authors, and wish the reader much joy and inspiration from studying it.

Jerusalem, January 2014

Preface

This book presents essentially the content of the CORE discussion papers (DP) 9420, 9421, and 9422 published as "Repeated Games," Parts A, B, and C in 1994. It may be appropriate to recall first the preface to those discussion papers:

> These notes represent work in progress, and far from its final form. An earlier version was circulated previously, and has been cited in various places. In view of this, we felt that the time had come to make it more widely available, in the form of discussion papers. We hope eventually to publish it in a more polished format. Remarks and suggestions are most welcome.
>
> Louvain-la-Neuve, June 1993

Unfortunately, the more polished published form was not realized, and the CORE discussion papers were out of print at some point. The objective of this book is to make this material accessible. Although several subsequent versions of this work have been available and were circulated, the material presented here is basically identical to that in the discussion papers with no intention to add new and recent material. We do, however, provide a more detailed presentation of the content, and in Appendix D we briefly introduce further developments after the DP version, along with the corresponding complementary bibliography.

Very sadly, this book is being published when Jean-François Mertens is no longer with us. He passed away on July 17, 2012. We obviously dedicate the book to him as a modest expression of our appreciation of his invaluable contributions to this project and to the underlying research in repeated games of incomplete information that led us to this book.

Sylvain Sorin
Shmuel Zamir
February 26, 2014

Acknowledgments

Our first acknowledgments are due to R. J. Aumann and the late M. Maschler, who, besides being among the founders of this research field of *repeated games with incomplete information*, were those who suggested this project to Shmuel Zamir and Jean-François Mertens more than thirty years ago . . . and from then on they persistently encouraged and even "pushed" the authors to terminate and publish it. As a matter of fact, it was Aumann's suggestion to publish it temporarily as a CORE discussion paper. Here we are happy to quote the following acknowledgment from the CORE discussion papers version:

> Support of CORE, and of numerous other institutions over the years, is gratefully acknowledged. So is Fabienne Henry's invaluable help in typing draft after draft.

We thank G. Mailath for suggesting this publication with Cambridge and for being patient enough to get the agreement and cooperation of the three of us. We are very grateful to B. von Stengel for helping us with some of the figures and to Ilan Nehama for his great help in dealing with the LaTeX software.

Finally, we are grateful to Mike Borns for proofreading the manuscript.

Sylvain Sorin
Shmuel Zamir
February 26, 2014

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Presentation of the Content

Part A collects basic results that will be used in the book.

In view of the large variety of games that are introduced and studied, it is necessary to present a general setup that will cover all cases (in the normal and extensive forms).

Chapter I deals with normal form games.

The first three sections (I.1, I.2, I.3) offer a comprehensive treatment of the minmax theorem. We start with an analysis of the case of pure strategies, basically Sion's theorem (Theorem I.1.1 in this volume) and some variants. We further treat the case of mixed strategies (Proposition I.1.9). The basic tool is the separation theorem, which is briefly studied. Then we present extensions corresponding to topological regularization (continuity, compactness), measurability requirements leading to the general "mixed form" (Theorem I.2.4), and purification of mixed strategies (Proposition I.2.7). Next we study the case of ordered fields (Theorem I.3.6), and the elementary finite approach is presented in I.3 Ex.[1]

The next section (I.4) is devoted to Nash equilibria (Theorem I.4.1), and several properties (manifold of equilibria, being semi-algebraic, fictitious play, etc.) are studied in I.4 Ex.

Chapter II defines extensive form games and treats successively the following topics:

Section II.1: The description of the extensive form, including the definition of pure, mixed, and behavioral strategies, linear games, and perfect recall (see also II.1 Ex.); Dalkey, Isbell, and Zermelo's theorems; and the measurable version of Kuhn's theorem (Theorem II.1.6).

Section II.2: The case of infinite games, first with perfect information, including Gale and Stewart's analysis and Martin's theorem (II.2.3) and then Blackwell's games (imperfect information) (Proposition II.2.8).

[1] "Ex." is short for exercise. Thus, "1.3 Ex." means the exercises in Chapter I, Section 3. See the paragraph on enumeration at the end of this Presentation (p. xxviii) for a more detailed explanation of the book's numbering system.

Section II.3: The notion of correlated equilibria, its properties (Aumann's theorem [Theorem II.3.2]), and several extensions: first, extensive form correlated equilibria, then communication equilibria (general formulation and properties; specific representation for finite games).

Section II.4: Games with vector payoffs and Blackwell's theorem (Theorem II.4.1).

The purpose of Chapter III is to study the interaction at the informational level, namely, the belief space.

We present a construction of the universal belief space (III.1) leading to Theorem III.1.1 (an alternative construction is in III.1 Ex. 2); its main properties (III.2 and III.3) such as belief subspaces, consistency, and relation with an information scheme (Theorem III.2.4) and the approximation (Theorem III.3.1).

In Section III.4 we describe the general model of games with incomplete information. First, we recover Harsanyi's model (Theorem III.4.1); then we prove, in the framework of two-person zero-sum games, regularity properties of the value. Proposition III.4.4 will be crucial for the recursive structure and the comparison of information (Proposition III.4.5). Further properties of extended approachability and convexity/concavity with respect to the information structure are then developed in Sections III.4.c and d.

Chapter IV is a presentation of the general class of repeated games.

After an exposition of the model (IV.1) including the (strong) notions of maxmin, minmax, and uniform equilibrium, we describe alternative representations (IV.2).

We then present the underlying recursive structure (IV.3) for the two-person zero-sum case leading to the basic Theorem IV.3.2.

The next section (IV.4) is devoted to supergames, that is, repeated games with complete information. We study Nash equilibria in the standard signaling case: uniform, discounted, and finite frameworks leading to Theorems IV.4.1, IV.4.2, and IV.4.4, respectively (perfect equilibria are treated in IV.4 Ex); then we give properties of uniform equilibrium payoffs for games with partial monitoring; and finally we study correlated and communication equilibrium payoffs.

Section IV.5 studies recursive games.

Part B treats the central results of the book: games with incomplete information (V and VI) and stochastic games (VII).

Chapter V deals with "repeated games with lack of information on one side." This corresponds to a two-person zero-sum repeated game where one of the players (Player 1) is fully informed and the other (Player 2) has no information.

Section V.1 proves concavity properties and the famous splitting procedure (Proposition V.1.2).

Section V.2 is devoted to the full monitoring case. We introduce the notion of posterior probabilities generated by the strategies and the bounds on the L^2 and L^1 variations of this martingale. Then we establish the basic lemma (V.2.3) relating the distance to the set of non-revealing strategies to the variation of the posteriors. The fundamental result is the $\mathsf{Cav}\, u$ Theorem V.2.10. Finally the approachability strategy of the noninformed player is described in Section V.2.c.

Section V.3 covers the general case of a signaling structure. We first describe the non-revealing game, then the extension of the $\mathsf{Cav}\, u$ Theorem V.3.3 and of the construction of an optimal strategy for player 2 in Section V.3.d. Sections V.3.e–3.i expose a general procedure for approachability in function spaces to be applied to the case of a continuum of types of Player 1.

Section V.4 develops the links between the recursive formula for the value, the maximal variation of the martingale, and the appearance of the normal law (Theorem V.4.1 and Theorem V.4.3).

Section V.5 studies the speed of convergence of v_n to its limit, first for the state-independent signaling case, then for state-dependent signaling.

Chapter VI covers "repeated games with lack of information on both sides." This corresponds to two-person zero-sum games where each player has some private information.

Section VI.2 presents the new $\mathsf{Cav}_{\mathrm{I}}$ and Vex_{II} operators and the extensions of the tools of Section V.2.

Section VI.3 studies the uniform approach and determines the maxmin and minmax of the infinite undiscounted game (Theorem VI.3.1).

Section VI.4 is concerned with the limit of the value v_n of the n-stage repeated game leading to the MZ system (Proposition VI.4.10).

Section VI.5 deals with further properties of the MZ equations.

Section VI.6 is devoted to the analysis of the speed of convergence of v_n to its limit, and Section VI.7 studies several examples in detail.

Chapter VII presents a general analysis of stochastic games.

Section VII.1 offers an analysis of the discounted case: first for zero-sum games (Propositions VII.1.4 and VII.1.5), then for (subgame perfect) equilibria in the n-person case, and for stationary strategies in the finite case (Proposition VII.1.7 and Theorem VII.1.8).

The algebraic approach is studied in Section VII.2.

Section VII.3 covers the main result dealing with the uniform approach (Theorem VII.3.1).

Section VII.4 considers two-person non-zero-sum absorbing games: we compare the different approaches in an example and prove the existence of equilibria (Theorem VII.4.6).

After Section VII.5, which is devoted to exercises (Shapley operator, $\lim v_n$, correlated equilibria, $\limsup$ payoffs, etc.), Section VII.6 offers a reminder about dynamic programming.

The last Part C presents further developments.

Chapter VIII is devoted to extensions and further results in a zero-sum framework.

Section VIII.1 deals with the case where the players have the same information and describes the reduction to absorbing games.

Section VIII.2 studies games with no signals. The analysis for the minmax and the maxmin is done through the construction of an auxiliary game in normal form that mimics the infinite game. The proof for $\lim v_n$ uses a sequence of games played by blocks.

Section VIII.3 introduces a game with lack of information on both sides with state-dependent signaling matrices. The analysis is conducted with the help of a family of auxiliary stochastic games and shows the link between the two fields of incomplete information and stochastic games.

Section VIII.4 is explicitly devoted to stochastic games with incomplete information and introduces new tools for the study of the minmax, maxmin, and $\lim v_n$.

Chapter IX is concerned with two-person non-zero-sum games with incomplete information on one side.

Section IX.1 gives an existence proof of uniform equilibrium in the case of two states of nature (Theorem IX.1.3), and a characterization of this set via “bi-martingales” (Theorem IX.1.4), which are explicitly studied in Section IX.2.

Section IX.3 introduces several communication devices and characterizes communication and “noisy channel” equilibrium payoffs.

Finally, Appendix A deals with analytic sets and Appendix B with historical notes.

Enumeration

Theorems, propositions, lemmas, corollaries, definitions, remarks, and examples are enumerated so that they can be easily referred to. The first part of the number is the chapter and section. Within each chapter two counters start from 1, 2, . . . , where theorems, propositions, lemmas, and corollaries use the same counter (thus for example, Proposition I.1.5 in Chapter I, Section 1, is followed by Theorem I.1.6). Similarly the second counter is for definitions, remarks, and examples (thus, for example, Remark III.2.9 in Chapter III, Section 2, is followed by Definition III.2.10).

Exercise enumeration is just 1, 2, . . . in each section. The reference to exercises is by indication of the chapter, section, and exercise number and part. For example: II.1, Ex. 9b is part b of Exercise 9 in Section 1 of Chapter II.

Figures are enumerated by chapter and counter (with no indication of section), e.g., Figure II.4.

Thanks

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