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Introduction

The use of statistical data analysis in physics means different things to different people. The reason for this is that most problems are different, and so someone concentrating on areas where the experimental data collected are relatively straightforward to analyse will naturally tend to use techniques that are simpler than those required for a more complicated problem or for a sparse data sample. Ultimately we all need to use statistical methods in order to translate data into some measure of a physical observable. This book will discuss a number of different concepts and techniques in order of increasing complexity. Before embarking on a discussion of statistics the remainder of this chapter introduces three common experimental problems encountered by students studying physics: (i) using a pendulum to measure acceleration due to gravity (Section 1.1), (ii) testing the validity of Ohm's law for a conductor (Section 1.2), and (iii) measuring the half-life of a radioactive isotope (Section 1.3). These examples rely on material covered in Chapters 4 through 7 and Chapter 9. Readers who appreciate the context of material in the remainder of this book may wish to skip forward to the next chapter.

1.1 Measuring g, the coefficient of acceleration due to gravity

The value of acceleration due to gravity (g typically reported in units of $\mathrm{m/s^2}$) changes slightly depending on where one makes the measurement. Precise maps of g are used in geological surveys as small local deviations in g may be indicative of mineral reserves, such as oil or gas. There are a number of ways of measuring this quantity and the one discussed here is the use of a swinging pendulum. This can be described as a simple harmonic oscillator with mass m, suspended on a string of length L. The period of oscillation T is given by $2\pi/\omega$, where ω is the angular frequency and is given by $\sqrt{g/L}$. Hence the corresponding period of oscillation for the pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}. \tag{1.1}$$

This is valid for small oscillations, as the small angle approximation of $\sin\theta \simeq \theta$ is used in deriving the relationship between T and g. By using Eq. (1.1) it is possible to estimate the acceleration due to the Earth's gravity from measurements of (i) the length L and (ii) the period of oscillation T of the pendulum. There is no dependence on the amplitude of oscillation (as long as the small-angle approximation remains valid) or the mass of the bob at the end of the pendulum. Equation (1.1) can be re-arranged as follows

$$g = \frac{4\pi^2 L}{T^2}, \tag{1.2}$$

so that one can directly obtain g from a single measurement or data point. Propagation of errors is discussed in Chapter 6 where in particular Eq. (6.11) can be used in order to determine the uncertainty on g, denoted by σ_g, given measurements and uncertainties on both T and L, where

$$\sigma_g^2 = \left(\frac{8\pi^2 L}{T^3}\right)^2 \sigma_T^2 + \left(\frac{4\pi^2}{T^2}\right)^2 \sigma_L^2. \tag{1.3}$$

Although this is a common experiment performed in many schools and undergraduate laboratories, many aspects of data analysis are required to fully appreciate issues that may arise with the measurement of g.

The most straightforward way to approach this problem is to measure the time it takes for a single complete oscillation to occur. From some maximum displacement one can release the pendulum bob and measure the time taken for the bob to reach back to where it started from. This measurement neglects any small effect arising from air resistance that may reduce the amplitude of oscillation slightly. There are several factors that one should consider when performing a measurement of g using this method.

- If a stopwatch is used to measure the period of oscillation, then there will be a significant uncertainty associated with starting and stopping the watch, relative to the period of time. For example if the oscillation period is of the order of a second, then the reaction time of the person starting and stopping the watch in quick succession will play an important role in the accuracy and precision of the time period measurement. One needs to ensure firstly that the measurement is accurate (i.e. that there is no systematic mistake made in timing), and that it is sufficiently precise. If the method of measuring the time period has an uncertainty of a second, then it would not be possible to make a useful measurement of time periods less than one second.

For example one can use the standard deviation on the ability of the experimenter to stop a stopwatch at a given count as a measure of the uncertainty of timing a particular event. As one has to both start and stop the stopwatch, twice this uncertainty can be ascribed as the uncertainty on time measurement. Using a trial of ten attempts to count to ten seconds on a stopwatch, the RMS deviation from that number was found to be 0.1 s. This was taken as the uncertainty on starting or stopping the watch. As both of these events have the same source of uncertainty they are taken to be correlated, hence twice that is used as the uncertainty on timing for an individual measurement of the oscillation i.e. ± 0.2 s.

- The relative uncertainty on L should also be small. For example, a measurement made with $L = 5.0 \pm 0.5$ cm would introduce a 10% uncertainty in Eq. (1.3), whereas this can be reduced to the percent level by increasing L to 50 cm. That in turn will increase T hence the relative precision on T using a given timing device as $T \propto \sqrt{L}$. So the experiment should be designed in such a way that L is sufficiently large so that it is not a dominant factor in the total uncertainty obtained for g, and mitigates contributions from timing.
- The relationship given in Eq. (1.1) is valid only for small amplitudes of oscillation, hence any measurement that deviates from a small amplitude of oscillation will result in a biased determination of g. The experimenter needs to understand this assumption and how the underlying approximation restricts the maximum displacement of the pendulum from the vertical position. For example, a length of $L = 50$ cm with an amplitude of oscillation of 10 cm results in a bias of -0.7% on the angle θ, which in turn introduces a small bias on g. As the effect is non-linear, an amplitude of oscillation of 20 cm results in a bias of -2.7% on the angle used in the approximation and so on. Therefore when starting to swing the pendulum, one should take care that the amplitude of oscillation is sufficiently small that the small angle approximation remains valid. A longer string length will help minimise this source of bias.

Having determined that the uncertainty on measurement timing is 0.2 s using a stopwatch, then a period of oscillation lasting 2 s will be determined to 10% (0.2 s /2.0 s). This in turn will limit the precision with which g can be determined as can be seen from the first term in Eq. (1.3). Table 1.1 shows several measurements of g made by using a stopwatch for timing and $L = 64.6 \pm 0.5$ cm. The measurement obtained using a single oscillation is $g = (9.7 \pm 2.4)$ m/s^2. The relative precision of this measurement is quite poor, only having determined g to 25%; however, it is possible to reduce the uncertainty on the measurement in several different

Table 1.1 *Measurements of g using the methodology described in the text.*

Number of measurements	Number of oscillations	g (m/s^2)
1	1	9.7 ± 2.4
10	1	10.2 ± 0.5
1	10	9.7 ± 0.3
10	10	9.7 ± 0.1

ways using the same experimental apparatus, but varying the experimental method slightly.

- One can make several measurements, and take the average of the ensemble as an estimate of the mean value of the period of oscillation, with the standard deviation corresponding to a measure of the spread in the data as the uncertainty on a measurement. The uncertainty on the mean value of the period from N individual measurements will scale by a factor $1/\sqrt{N}$. This means that one can quickly make improvements on a single measurement, by making several subsequent measurements, but soon the increase in precision obtained by making an additional single measurement will become small. This assumes that each trial measurement is made under identical conditions, and neglects any systematic mistakes in measuring L or T.
- Another way to improve the precision would be to measure the time taken for several periods of oscillation. The uncertainty on a single period of oscillation derived from a measurement of ten oscillations is $\sigma_T = \sigma_{10T}/10$, i.e. the uncertainty on the measurement is spread equally across each oscillation that occurred within the measurement, and one can effectively reduce the statistical uncertainty on T by a factor of ten.
- The period of oscillation achievable for a given setup depends on the length L. Given that g is a constant for a given laboratory, the longer the pendulum, the longer the period of oscillation. So one can increase L for a given measurement method in order to reduce the relative precision on the measured value of g, within practical limitations of the experimental apparatus.

The precision obtained for the measured value of g can be improved further by averaging a number of measurements made of multiple oscillations at a time, for the longest pendulum length L allowable by the the apparatus, i.e. by taking into account the three previous considerations. Table 1.1 summarises the results of several sets of measurements of g and in particular illustrates the potential for improvements obtained by making multiple measurements, and by measuring several periods of oscillation. One can see from the results shown in the table that while making several measurements of a given number of oscillations and averaging those results gives an improvement in precision over a single measurement of g, the most effective way to improve the precision using this apparatus is to increase the number of oscillations counted in order to estimate T. By applying a basic understanding of statistical data analysis to this problem it has been possible to refine the experimental method in order to determine the value of g with a relative precision of 1%, compared with the 25% relative precision obtained initially. For a measurement with a 1% statistical precision, a systematic bias of −0.7% from the use of the small angle approximation would be a concern. The underlying techniques used here are discussed in Chapters 4 and 6 and the reader may wish to re-read this example once they have reached the end of Chapter 6.

1.2 Verification of Ohm's law

This example builds on some of the techniques discussed above. We are surrounded by electronic devices in the modern world. One of the fundamental laws related to electronics is that of Ohm's law: the voltage V across an Ohmic conductor is proportional to the electric current I passing through it. The constant of proportionality is called the resistance of the conductor, and components called resistors that are made out of Ohmic conductors pervade our lives in numerous ways. The electronic circuits in your mobile phone, television, and computer have hundreds of resistors in them, and without the simple resistor those devices would cease to function. The underlying principle of Ohm's law underpinning the concept of the resistor is

$$V = IR. \tag{1.4}$$

Given the form of this relationship it is possible to take a single measurement of V and I and subsequently compute an estimate of R. This gives the resistance of the conductor for a given data point, but does not allow the experimenter to verify if the conductor is Ohmic.

The measurement of a single data point depends on the precision with which the voltage and current were measured. As an example we can consider measurements made on a 12 000 Ω resistor using a hand-held digital multi-meter (DMM) to determine the voltage across the resistor, and a precision DMM to determine the current passing through the resistor. In this case the limiting factor is the voltage measurement, which was made with a precision of 0.09% and an additional

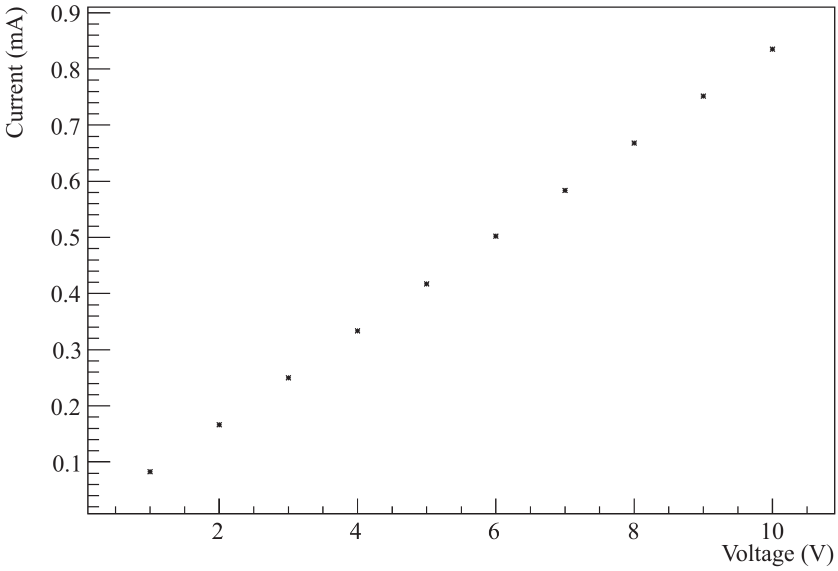

Figure 1.1 The distribution of current (I) passing through a 12 000 Ω resistor versus the voltage (V) across it.

uncertainty of two in the last digit read off the DMM.[1] The current was measured with a more precise device, so it is sufficient to assume that the uncertainty on a single measurement of R discussed here is dominated by the contribution from the voltage measurement. The resistance measured, with a current of 0.8349 mA passing through it and a voltage of 10.0 V across it, is $11\,977 \pm 35\ \Omega$.

If one assumes that the conductor was known to be Ohmic, which is reasonable for a resistor, then one could simply average the values of resistance obtained for a number of different measurements. As with the example of measuring g above, the mean and standard deviation of the data could be used to determine an estimate of the resistance and uncertainty of the component under study (see Chapter 4). The mean resistance computed for a 12 000 Ω resistor as obtained from the ten data points shown in Figure 1.1 is 11 996 Ω, with a standard deviation of 30 Ω. This is in good agreement with the estimate of the central value from a single measurement. However, in general the precision from ten measurements should be $\sqrt{10}$ times better (smaller) than that of a single measurement. As this is not the case with these data one might worry that there could be systematic effect (such as linearity of a measuring device, or temperature of the laboratory) that is not being taken into

[1] DMMs need to be regularly calibrated in order to ensure that measurements made are accurate, and that the precisions of measurements as quoted in their instruction manuals are valid. There may also be systematic effects that one has to consider, such as temperature dependences on a reported reading. The accuracy and limitations of a measuring device used needs to be understood in order to ensure that the experimenter can correctly interpret the data recorded.

account. More generally if it is not known if the component is an Ohmic conductor one could plot I against V and determine if Eq. (1.4) was valid or not. Figure 1.1 shows data recorded for a 12 000 Ω resistor over a voltage range of 1–10 V. The linear relationship between I and V is evident by eye.

The value of R measured for this resistor could be determined more precisely by fitting the data as described in Chapter 9. A χ^2 fit to V/I vs V assuming Ohm's law yields $R = 11\,996 \pm 10\,\Omega$, in good agreement with previous determinations. The χ^2 for this fit is 8.87 for nine degrees of freedom, which corresponds to a probability of $P(\chi^2, \nu) = P(8.87, 9) = 0.45$ which is quite reasonable (see Chapter 5). If one looks at the data in Figure 1.1, one might wonder if the value of R might be changing slightly with voltage. This can be investigated by taking data over a wide range of V, or using the existing data and changing the model used in the fit. The next simplest model to Ohm's law would be to introduce a linear variation in the resistance as a function of voltage. A χ^2 fit to V/I vs V assuming a relationship of the form of $y = mx + R$ to allow for a possible change in resistance as a function of voltage yields $R = 12\,037 \pm 20\,\Omega$ and $m = -7.5 \pm 3.3$ with $\chi^2 = 3.75$ for eight degrees of freedom. The slope m obtained is consistent with zero within uncertainties, and the value of R obtained is consistent with the previous determination, but has a larger uncertainty. The probability of this fit is quite reasonable, $P(\chi^2, \nu) = P(3.75, 8) = 0.878$. While the result of the second fit is more probable than the first, there is insufficient motivation to support the hypothesis that the behaviour observed deviates from Ohm's law, as the slope coefficient obtained from the second fit is consistent with zero. Examining the data using techniques such as this allows one to go beyond a qualitative inspection of graphs and to quantify the underlying physical observables. Chapter 9 discusses several different fitting techniques that could have been applied to this problem.

Looking closely at the data it appears that the value of R computed for the first data point is slightly higher than the rest, and as a result will be the main contributor to the value of the χ^2 obtained when testing Ohm's law. While this data point is reasonable one might be concerned about the integrity of the data in general. In such a situation there are several options that one might naturally consider: (i) taking more data points at lower values of V, (ii) repeating the full set of measurements, and (iii) studying the specifications of the measuring devices to ensure that the uncertainties have been correctly interpreted when making each of the measurements.

1.3 Measuring the half-life of an isotope

A common undergraduate experiment involves the determination of the decay constant (λ) or half-life ($t_{1/2} = \ln 2/\lambda$) of a radioactive isotope. The number of

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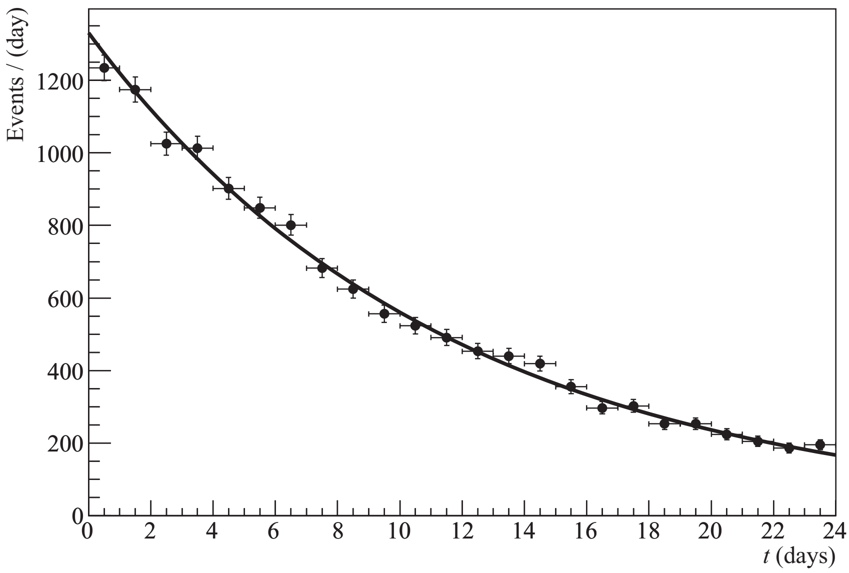

Figure 1.2 The distribution of events (count rate) expected in an experiment studying the radioactive decay of ^{131}I.

radioactive nuclei at a given time t is

$$N(t) = N_0 e^{-\lambda t}, \tag{1.5}$$

where N_0 is the initial number of nuclei in the sample (at time $t = 0$). The rate of decay of a radioactive isotope is given by

$$\frac{dN}{dt} = -\lambda N(t). \tag{1.6}$$

The decay constant λ introduced above can be understood as the rate of change of the number of radioactive nuclei of a given type with respect to time elapsed. This quantity is related to the aptly called half-life of an isotope. The ***half-life*** is the time taken for the number of radioactive nuclei to reduce by one half with respect to a given time. It follows from Eq. (1.5) that after one half-life

$$\frac{N(t)}{N_0} = e^{-\lambda t_{1/2}} = \frac{1}{2}, \tag{1.7}$$

hence $t_{1/2} = \ln 2/\lambda$.

It is possible to measure the decay constant of a radioactive isotope by studying the number of counts observed in a radiation detector, for example a Geiger–Müller tube, as a function of time. Each count corresponds to the detection of the decay product of a radioactive nuclei disintegrating. From the expected time dependence one can extract the decay constant and in turn convert this into a measure of the half-life of the isotope. Figure 1.2 shows the result of a simulation of the count rate expected as a function of time for an ^{131}I source, which produces β radiation with a half-life of 8.02 days. This isotope is used in a number of medical physics applications. The simulation neglects background, which can be measured in the absence of the source and subtracted from data, or alternatively taken into account at the same time as extracting λ from the data. Background from the naturally occurring radioactivity of the experimental environment is expected to be uniform in time. There are a number of possible sources of background including cosmic rays and natural radioactivity from rocks or other materials in the environment, e.g. areas rich in granite often have elevated levels of radon gas which is radioactive, and thus a marginally higher than normal level of background radiation. While this background varies from location to location, for a given laboratory the background rate should be constant.

The ^{131}I signal shown in the figure follows the radioactive decay law of Eq. (1.5), with an exponentially decreasing count rate. The data are displayed as a binned histogram (see Chapter 4), with Poisson error bars[2] on the content of each of the bins (see Chapters 6 and 7). As the number of entries in a given bin (so the count rate) decrease, so the relative size of the error on the count rate increases. The data are fitted using an un-binned extended maximum likelihood fit as described in Chapter 9 using a model corresponding to a signal exponentially decaying with time (i.e. the ^{131}I). So while the data are visually displayed in bins of counts in any given day, the individual time of a count is used in the fit to data, and the binning is essentially for display purposes only. The results of this fit to data will be the values and uncertainties of the signal yield (so how many signal counts were recorded), and the decay constant measured for the ^{131}I sample. Given the relationship between the decay constant and half-life, it is possible to convert the value of λ obtained in the fit to data into a measurement of the half-life using the error propagation formalism introduced in Chapter 6. In order to avoid having to translate the fitted parameter into the half-life, one can re-parameterise the likelihood function to fit for $t_{1/2}$ directly.

This is just one way to analyse the data, instead of performing an un-binned extended maximum likelihood fit to the data, we could have binned the data before fitting. On doing this some information is lost, but if there are sufficient data to analyse, any loss in precision would be negligible. Another alternative would be to perform a χ^2 fit to data (as was used for the example of studying Ohmic conductors discussed above). Yet another way to analyse the data would be to perform a linear least squares regression analysis. In order to do this it is convenient to convert the data into a form where one has a linear relationship between the quantities plotted on the ordinate and abscissa. Given that the measured count rate is

[2] The process of detecting the products of a radioactive decay is described by a Poisson probability distribution (see Chapter 5 for more details), and so the uncertainty ascribed to the content of a given bin is Poisson.

given by

$$N(t) = N_0 e^{-\lambda t} + N_{bg} U(t), \tag{1.8}$$

where $U(t)$ is a uniform probability density function (see Appendix B) describing some number of background events N_{bg}, it follows that the background corrected rate $N'(t) = N(t) - N_{bg}$ satisfies

$$\ln N'(t) = \ln N_0 - \lambda t. \tag{1.9}$$

Hence, one can determine the decay constant from the slope of the logarithm $\ln N'(t)$ vs time, and one can compute the value and uncertainty on λ without the need for sophisticated minimisation techniques that underpin χ^2 or maximum-likelihood fitting approaches. In addition, the initial total number of radioactive nuclei can be computed from the constant term, $\ln N_0$, should this be of interest. All of the techniques mentioned here are described in Chapter 9. The reader may wish to re-examine this example once they have reached the end of Chapter 9 in order to reflect on the approach taken, and on how some of the alternate techniques mentioned above may be applied to this problem.

A number of physical processes obey an exponential decay (or growth) law where the data can be analysed in a similar way to that described here. For example measurement of the lifetime of the decay of a sub-atomic particle, such as muons found in cosmic rays, uses the same data analysis technique(s) as the decay constant or half-life analysis discussed here. Similarly data obtained in order to determine the attenuation of light in a transparent material, or radioactive particles passing through different thicknesses of shielding follow an exponential attenuation law and can be analysed using one of the approaches described here. The basis for this type of data analysis can also be adapted in order to address more complicated situations where the physical model involves more parameters (λ, N_0 and N_{bg} are physical parameters associated with this particular problem), more than one signal or background component, or indeed more dimensions (the only dimension considered in this example is time) containing information that can be used to distinguish between the components. If the number of dimensions one wishes to analyse becomes large, then it may be appropriate for the analyst to investigate the use of a multivariate algorithm to distinguish between signal and background samples of events. Chapter 10 introduces a number of algorithms that can be used for such problems.

1.4 Summary

Students encountering statistical methods in an undergraduate laboratory course often express a lack of enthusiasm for the relevance of the topic. In an attempt