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Topics in Critical Point Theory

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To my wife Champa.
K.P.

To my wife, Deborah, our children,
our grandchildren (twenty five so far),
our great grandchildren (fifteen so far),
and our extended family.
May they all enjoy many happy years.
M.S.

Contents

Preface

Critical point theory has become a very powerful tool for solving many problems. The theory has enjoyed significant development over the past several years. The impetus for this development is the fact that many new problems could not be solved by the older theory.

There have been several excellent books written on critical point theory from various points of view; see, e.g., Berger [19], Zeidler [161], Rabinowitz [129], Mawhin and Willem [91], Chang [29, 30], Ghoussoub [56], Ambrosetti and Prodi [8], Willem [158], Chabrowski [26], Dacorogna [36], and Struwe [153] (see also Schechter [143, 144, 147], Zou and Schechter [163], and Perera *et al.* [113]). In this book we present more recent developments in the subject that do not seem to be covered elsewhere, including some results of the authors dealing with nonstandard linking geometries and sandwich pairs.

Chapter 1 is a brief review of Morse theory in Banach spaces. We prove the first and second deformation lemmas under the Cerami compactness condition. As the variational functionals associated with applications given later in the book will only be C^1, we discuss critical groups of C^1-functionals. We include discussions on minimizers, nontrivial critical points, mountain pass points, and the three critical points theorem. We also give a generalized notion of local linking that yields a nontrivial critical group, which will be applied to problems with jumping nonlinearities in Chapter 5. We close the chapter with a recent result of Perera [110] on nontrivial critical groups in p-Laplacian problems.

Chapter 2 is on linking. We say that subsets A, B of a Banach space E link if every C^1-functional G on E satisfying

$$-\infty < a := \sup_A G \leqslant \inf_B G =: b < +\infty,$$

and a suitable compactness condition, has a critical point u with $G(u) \geqslant b$. There are three main notions of linking; homological, homotopical, and a more recent one introduced by Schechter and Tintarev [148]. We discuss all three and show that

$$\begin{matrix}\text{homological} \\ \text{linking}\end{matrix} \implies \begin{matrix}\text{homotopical} \\ \text{linking}\end{matrix} \implies \begin{matrix}\text{Schechter–Tintarev} \\ \text{linking.}\end{matrix}$$

We also discuss some results of Schechter and Tintarev [148] on pairs of critical points produced by linking subsets and some results on their critical groups due to Perera [104]. We close with some recent results of the authors [118] that give critical points with nontrivial critical groups under nonstandard geometrical assumptions that do not involve a finite-dimensional closed loop.

Chapter 3 contains applications of the Morse theoretic and linking methods of the first two chapters to semilinear elliptic boundary value problems. We discuss the local nature of critical groups and critical groups at zero. We consider asymptotically linear problems and problems with concave nonlinearities, and obtain multiple nontrivial solutions using our nonstandard linking theorems.

Chapter 4 considers the Fučík spectrum in an abstract operator setting, that includes many concrete problems arising in applications as special cases. We construct the minimal and maximal curves of the spectrum locally near the points where it intersects the main diagonal of the plane. We give a sufficient condition for the region between them to be nonempty, and show that it is free of the spectrum in the case of a simple eigenvalue. Finally we compute the critical groups in various regions separated by these curves. We compute them precisely in Type I regions, and prove a shifting theorem that gives a finite-dimensional reduction to the null manifold for Type II regions.

Chapter 5 is a continuation of the previous chapter that considers problems with jumping nonlinearities in the same abstract framework. We discuss compactness and critical groups at infinity and zero. We compute critical groups in both resonant and nonresonant problems. This allows us to establish solvability in Type I regions, and obtain nontrivial solutions for nonlinearities crossing a curve of the Fučík spectrum constructed in Chapter 4.

Chapter 6 is on sandwich pairs. We say that a pair of subsets A, B of a Banach space E is a sandwich pair if every C^1-functional G on E satisfying

$$-\infty < b := \inf_B G \leqslant \sup_A G =: a < +\infty,$$

and a suitable compactness condition, has a critical point u with $b \leqslant G(u) \leqslant a$. We construct a very general class of sandwich pairs with wide applicability using a family of flows on E and the Fadell–Rabinowitz cohomological index. We use a special case of this where the sandwich pair is a certain pair of cones to solve p-Laplacian problems. We also solve anisotropic p-Laplacian systems using another special case with a curved sandwich pair made up of certain orbits of an associated group action on a product of Sobolev spaces.