1

Morse theory

1.1 Introduction

The purpose of this chapter is to introduce the reader to Morse theoretic methods used in variational problems. General references are Milnor [93], Mawhin and Willem [91], Chang [29], and Benci [17]; see also Perera *et al.* [113]. We begin by briefly collecting some basic results of Morse theory. These include the Morse inequalities, Morse lemma and its generalization splitting lemma, the shifting theorem of Gromoll and Meyer, and the handle body theorem. Results that are needed later in the text will be proved in subsequent sections.

Let G be a real-valued function defined on a real Banach space E. We say that G is Fréchet differentiable at $u \in E$ if there is an element $G'(u)$ of the dual E', called the Fréchet derivative of G at u, such that

$$G(u+v) = G(u) + \big(G'(u), v\big) + \mathrm{o}(\|v\|) \quad \text{as} \quad v \to 0 \quad \text{in} \quad E,$$

where $(\cdot, \cdot)$ is the duality pairing. The functional G is continuously Fréchet differentiable on E, or belongs to the class $C^1(E, \mathbb{R})$, if G' is defined everywhere and the map $E \to E',\ u \mapsto G'(u)$ is continuous. We assume that $G \in C^1(E, \mathbb{R})$ for the rest of the chapter. Replacing G with $G - G(0)$ if necessary, we may also assume that $G(0) = 0$. The functional G is called even if

$$G(-u) = G(u) \quad \forall u \in E.$$

Then G' is odd, i.e.,

$$G'(-u) = -G'(u) \quad \forall u \in E.$$

We say that u is a critical point of G if $G'(u) = 0$. A value c of G is a critical value if there is a critical point u with $G(u) = c$, otherwise it is a regular value.

We use the standard notations

$$
\begin{aligned}
&G_a = \{u \in E : G(u) \geqslant a\}, \qquad G^b = \{u \in E : G(u) \leqslant b\},\\
&G_a^b = G_a \cap G^b,\\
&K = \{u \in E : G'(u) = 0\}, \qquad \widehat{E} = E \setminus K,\\
&K_a^b = K \cap G_a^b, \qquad K^c = K_c^c
\end{aligned}
$$

for the various superlevel, sublevel, critical, and regular sets of G.

It is usually necessary to assume some sort of a "compactness condition" when seeking critical points of a functional. The following condition was originally introduced by Palais and Smale [101]: G satisfies the Palais–Smale compactness condition at the level c, or $(\mathrm{PS})_c$ for short, if every sequence $(u_j) \subset E$ such that

$$G(u_j) \to c, \quad G'(u_j) \to 0,$$

called a $(\mathrm{PS})_c$ sequence, has a convergent subsequence; G satisfies (PS) if it satisfies $(\mathrm{PS})_c$ for every $c \in \mathbb{R}$, or equivalently, if every sequence such that $G(u_j)$ is bounded and $G'(u_j) \to 0$, called a (PS) sequence, has a convergent subsequence. The following weaker version was introduced by Cerami [25]: G satisfies the Cerami condition at the level c, or $(\mathrm{C})_c$ for short, if every sequence such that

$$G(u_j) \to c, \quad \left(1 + \|u_j\|\right) G'(u_j) \to 0,$$

called a $(\mathrm{C})_c$ sequence, has a convergent subsequence; G satisfies (C) if it satisfies $(\mathrm{C})_c$ for every c, or equivalently, if every sequence such that $G(u_j)$ is bounded and $\left(1 + \|u_j\|\right) G'(u_j) \to 0$, called a (C) sequence, has a convergent subsequence. This condition is weaker since a $(\mathrm{C})_c$ (resp. (C)) sequence is clearly a $(\mathrm{PS})_c$ (resp. (PS)) sequence also. The limit of a $(\mathrm{PS})_c$ (resp. (PS)) sequence is in K^c (resp. K) since G and G' are continuous. Since any sequence in K^c is a $(\mathrm{C})_c$ sequence, it follows that K^c is a compact set when $(\mathrm{C})_c$ holds.

Some of the essential tools for locating critical points are the deformation lemmas, which allow to lower sublevel sets of a functional, away from its critical set. The main ingredient in their proofs is a suitable negative pseudo-gradient flow, a notion due to Palais [103]: a pseudo-gradient vector field for G on $\widehat{E}$ is a locally Lipschitz continuous mapping $V : \widehat{E} \to E$ satisfying

$$\|V(u)\| \leqslant \|G'(u)\|, \quad 2\left(G'(u), V(u)\right) \geqslant \left(\|G'(u)\|\right)^2 \quad \forall u \in \widehat{E}.$$

Such a vector field exists, and may be chosen to be odd when G is even.

The first deformation lemma provides a local deformation near a (possibly critical) level set of a functional.

Lemma 1.1.1 (first deformation lemma) *If $c \in \mathbb{R}$, C is a bounded set containing K^c, $\delta, k > 0$, and G satisfies $(\mathrm{C})_c$, then there are an $\varepsilon_0 > 0$ and, for each $\varepsilon \in (0, \varepsilon_0)$, a map $\eta \in C(E \times [0,1], E)$ satisfying*

(i) $\eta(\cdot, 0) = id_E$,
(ii) $\eta(\cdot, t)$ *is a homeomorphism of E for all $t \in [0,1]$,*
(iii) $\eta(\cdot, t)$ *is the identity outside $A = G_{c-2\varepsilon}^{c+2\varepsilon} \backslash N_{\delta/3}(C)$ for all $t \in [0,1]$,*
(iv) $\|\eta(u,t) - u\| \leqslant \big(1 + \|u\|\big)\, \delta/k \quad \forall (u,t) \in E \times [0,1]$,
(v) $G(\eta(u, \cdot))$ *is nonincreasing for all $u \in E$,*
(vi) $\eta(G^{c+\varepsilon} \backslash N_\delta(C), 1) \subset G^{c-\varepsilon}$.

When G is even and C is symmetric, η may be chosen so that $\eta(\cdot, t)$ is odd for all $t \in [0,1]$.

The first deformation lemma under the $(\mathrm{PS})_c$ condition is due to Palais [102]; see also Rabinowitz [126]. The proof under the $(\mathrm{C})_c$ condition was given by Cerami [25] and Bartolo *et al.* [13]. The particular version given here will be proved in Section 1.3.

The second deformation lemma implies that the homotopy type of sublevel sets can change only when crossing a critical level.

Lemma 1.1.2 (second deformation lemma) *If $-\infty < a < b \leqslant +\infty$ and G has only a finite number of critical points at the level a, has no critical values in (a, b), and satisfies $(\mathrm{C})_c$ for all $c \in [a, b] \cap \mathbb{R}$, then G^a is a deformation retract of $G^b \backslash K^b$, i.e. there is a map $\eta \in C((G^b \backslash K^b) \times [0,1], G^b \backslash K^b)$, called a deformation retraction of $G^b \backslash K^b$ onto G^a, satisfying*

(i) $\eta(\cdot, 0) = id_{G^b \backslash K^b}$,
(ii) $\eta(\cdot, t)|_{G^a} = id_{G^a} \quad \forall t \in [0,1]$,
(iii) $\eta(G^b \backslash K^b, 1) = G^a$.

The second deformation lemma under the $(\mathrm{PS})_c$ condition is due to Rothe [135], Chang [28], and Wang [157]. The proof under the $(\mathrm{C})_c$ condition can be found in Bartsch and Li [14], Perera and Schechter [119], and in Section 1.3.

In Morse theory the local behavior of G near an isolated critical point u is described by the sequence of critical groups

$$C_q(G, u) = H_q(G^c \cap U, G^c \cap U \backslash \{u\}), \quad q \geqslant 0$$

where $c = G(u)$ is the corresponding critical value, U is a neighborhood of u, and H_* denotes singular homology. They are independent of the choice of U by the excision property.

For example, if u is a local minimizer, $C_q(G, u) = \delta_{q0}\,\mathcal{G}$ where δ is the Kronecker delta and $\mathcal{G}$ is the coefficient group. A critical point u with $C_1(G, u) \neq 0$ is called a mountain pass point.

Let $-\infty < a < b \leqslant +\infty$ be regular values and assume that G has only isolated critical values $c_1 < c_2 < \cdots$ in (a, b), with a finite number of critical points at each level, and satisfies $(\mathrm{PS})_c$ for all $c \in [a, b] \cap \mathbb{R}$. Then the Morse type numbers of G with respect to the interval (a, b) are defined by

$$M_q(a, b) = \sum_i \operatorname{rank} H_q(G^{a_{i+1}}, G^{a_i}), \quad q \geqslant 0$$

where $a = a_1 < c_1 < a_2 < c_2 < \cdots$. They are independent of the a_i by the second deformation lemma, and are related to the critical groups by

$$M_q(a, b) = \sum_{u \in K_a^b} \operatorname{rank} C_q(G, u).$$

Writing $\beta_j(a, b) = \operatorname{rank} H_j(G^b, G^a)$, we have the following.

Theorem 1.1.3 (Morse inequalities) *If there is only a finite number of critical points in G_a^b, then*

$$\sum_{j=0}^{q} (-1)^{q-j} M_j \geqslant \sum_{j=0}^{q} (-1)^{q-j} \beta_j, \quad q \geqslant 0,$$

and

$$\sum_{j=0}^{\infty} (-1)^j M_j = \sum_{j=0}^{\infty} (-1)^j \beta_j$$

when the series converge.

Critical groups are invariant under homotopies that preserve the isolatedness of the critical point; see Rothe [134], Chang and Ghoussoub [27], and Corvellec and Hantoute [32].

Theorem 1.1.4 *If $G_t,\ t \in [0, 1]$ is a family of C^1-functionals on E satisfying (PS), u is a critical point of each G_t, and there is a closed neighborhood U such that*

(i) U contains no other critical points of G_t,
(ii) the map $[0, 1] \to C^1(U, \mathbb{R}),\ t \mapsto G_t$ is continuous,

then $C_*(G_t, u)$ are independent of t.

When the critical values are bounded from below and G satisfies (C), the global behavior of G can be described by the critical groups at infinity introduced by Bartsch and Li [14]

$$C_q(G, \infty) = H_q(E, G^a), \quad q \geqslant 0$$

where a is less than all critical values. They are independent of a by the second deformation lemma and the homotopy invariance of the homology groups.

For example, if G is bounded from below, $C_q(G, \infty) = \delta_{q0}\, \mathcal{G}$. If G is unbounded from below, $C_q(G, \infty) = \widetilde{H}_{q-1}(G^a)$ where $\widetilde{H}$ denotes the reduced groups.

Proposition 1.1.5 *If $C_q(G, \infty) \neq 0$ and G has only a finite number of critical points and satisfies* (C)*, then G has a critical point u with $C_q(G, u) \neq 0$.*

The second deformation lemma implies that $C_q(G, \infty) = C_q(G, 0)$ if $u = 0$ is the only critical point of G, so G has a nontrivial critical point if $C_q(G, 0) \neq C_q(G, \infty)$ for some q.

Now suppose that E is a Hilbert space $(H, (\cdot, \cdot))$ and $G \in C^2(H, \mathbb{R})$. Then the Hessian $A = G''(u)$ is a self-adjoint operator on H for each u. When u is a critical point the dimension of the negative space of A is called the Morse index of u and is denoted by $m(u)$, and $m^*(u) = m(u) + \dim \ker A$ is called the large Morse index. We say that u is nondegenerate if A is invertible. The Morse lemma describes the local behavior of the functional near a nondegenerate critical point.

Lemma 1.1.6 (Morse lemma) *If u is a nondegenerate critical point of G, then there is a local diffeomorphism ξ from a neighborhood U of u into H with $\xi(u) = 0$ such that*

$$G(\xi^{-1}(v)) = G(u) + \frac{1}{2}\,(Av, v)\,, \quad v \in \xi(U).$$

Morse lemma in $\mathbb{R}^n$ was proved by Morse [95]. Palais [102], Schwartz [149], and Nirenberg [98] extended it to Hilbert spaces when G is C^3. Proof in the C^2 case is due to Kuiper [64] and Cambini [23].

A direct consequence of the Morse lemma is the following theorem.

Theorem 1.1.7 *If u is a nondegenerate critical point of G, then*

$$C_q(G, u) = \delta_{qm(u)}\, \mathcal{G}.$$

The handle body theorem describes the change in topology as the level sets pass through a critical level on which there are only nondegenerate critical points.

Theorem 1.1.8 (handle body theorem) *If c is an isolated critical value of G for which there are only a finite number of nondegenerate critical points u_i, $i = 1, \dots, k$, with Morse indices $m_i = m(u_i)$, and G satisfies* (PS)*, then there are an $\varepsilon > 0$ and homeomorphisms φ_i from the unit disk D^{m_i} in $\mathbb{R}^{m_i}$ into H such that*

$$G^{c-\varepsilon} \cap \varphi_i(D^{m_i}) = G^{-1}(c - \varepsilon) \cap \varphi_i(D^{m_i}) = \varphi_i(\partial D^{m_i})$$

and $G^{c-\varepsilon} \cup \bigcup_{i=1}^k \varphi_i(D^{m_i})$ is a deformation retract of $G^{c+\varepsilon}$.

The references for Theorems 1.1.3, 1.1.7, and 1.1.8 are Morse [95], Pitcher [124], Milnor [93], Rothe [132, 133, 135], Palais [102], Palais and Smale [101], Smale [151], Marino and Prodi [89], Schwartz [149], Mawhin and Willem [91], and Chang [29].

The splitting lemma generalizes the Morse lemma to degenerate critical points. Assume that the origin is an isolated degenerate critical point of G and 0 is an isolated point of the spectrum of $A = G''(0)$. Let $N = \ker A$ and write $H = N \oplus N^\perp$, $u = v + w$.

Lemma 1.1.9 (splitting lemma) *There are a ball $B \subset H$ centered at the origin, a local homeomorphism ξ from B into H with $\xi(0) = 0$, and a map $\eta \in C^1(B \cap N, N^\perp)$ such that*

$$G(\xi(u)) = \frac{1}{2}(Aw, w) + G(v + \eta(v)), \quad u \in B.$$

Splitting lemma when A is a compact perturbation of the identity was proved by Gromoll and Meyer [57] for $G \in C^3$ and by Hofer [60] in the C^2 case. Mawhin and Willem [90, 91] extended it to the case where A is a Fredholm operator of index zero. The general version given here is due to Chang [29].

A consequence of the splitting lemma is the following.

Theorem 1.1.10 (shifting theorem) *We have*

$$C_q(G, 0) = C_{q-m(0)}(G|_{\mathcal{N}}, 0) \quad \forall q$$

where $\mathcal{N} = \xi(B \cap N)$ is the degenerate submanifold of G at 0*.*

The shifting theorem is due to Gromoll and Meyer [57]; see also Mawhin and Willem [91] and Chang [29].

Since $\dim \mathcal{N} = m^*(0) - m(0)$, the shifting theorem gives us the following Morse index estimates when there is a nontrivial critical group.

Corollary 1.1.11 *If $C_q(G, 0) \neq 0$, then*

$$m(0) \leqslant q \leqslant m^*(0).$$

It also enables us to compute the critical groups of a mountain pass point of nullity at most one.

Corollary 1.1.12 *If u is a mountain pass point of G and* $\dim \ker G''(u) \leqslant 1$, *then*

$$C_q(G, u) = \delta_{q1}\, \mathcal{G}.$$

This result is due to Ambrosetti [4, 5] in the nondegenerate case and to Hofer [60] in the general case.

Shifting theorem also implies that all critical groups of a critical point with infinite Morse index are trivial, so the above theory is not suitable for studying strongly indefinite functionals. An infinite-dimensional Morse theory particularly well suited to deal with such functionals was developed by Szulkin [155]; see also Kryszewski and Szulkin [63].

The following important perturbation result is due to Marino and Prodi [88]; see also Solimini [152].

Theorem 1.1.13 *If some critical value of G has only a finite number of critical points u_i, $i = 1, \dots, k$ and $G''(u_i)$ are Fredholm operators, then for any sufficiently small $\varepsilon > 0$ there is a C^2-functional G_ε on H such that*

(*i*) $\|G_\varepsilon - G\|_{C^2(H)} \leqslant \varepsilon$,
(*ii*) $G_\varepsilon = G$ *in* $H \setminus \bigcup_{i=1}^k B_\varepsilon(u_j)$,
(*iii*) G_ε *has only nondegenerate critical points in* $B_\varepsilon(u_j)$ *and their Morse indices are in* $[m(u_i), m^*(u_i)]$,
(*iv*) G *satisfies* (PS) $\implies$ G_ε *satisfies* (PS).

Here

$$B_r(u_0) = \{u \in H : \|u - u_0\| \leqslant r\}$$

is the closed ball of radius r centered at u_0. We will write B_r for $B_r(0)$ in the sequel.

Returning to the setting of a C^1-functional on a Banach space E, in many applications G has the trivial critical point $u = 0$ and we are interested in finding others. The notion of a local linking introduced by Li and Liu [70, 83] is useful for obtaining nontrivial critical points under various assumptions on the behavior of G at infinity; see also Brezis and Nirenberg [21] and Li and Willem [72]. Assume that the origin is a critical point of G with $G(0) = 0$. We say that G has a local linking near the origin if there is a direct sum decomposition $E = N \oplus M$, $u = v + w$ with N finite dimensional such that

$$\begin{cases} G(v) \leqslant 0, & v \in N, \ \|v\| \leqslant r \\ G(w) > 0, & w \in M, \ 0 < \|w\| \leqslant r \end{cases}$$

for sufficiently small $r > 0$. Liu [82] showed that this yields a nontrivial critical group at the origin.

Proposition 1.1.14 *If G has a local linking near the origin with $\dim N = d$ and the origin is an isolated critical point, then $C_d(G, 0) \neq 0$.*

The following alternative obtained in Perera [106] gives a nontrivial critical point with a nontrivial critical group produced by a local linking.

Theorem 1.1.15 *If G has a local linking near the origin with $\dim N = d$, $H_d(G^b, G^a) = 0$ where $-\infty < a < 0 < b \leqslant +\infty$ are regular values, and G has only a finite number of critical points in G_a^b and satisfies $(\mathrm{C})_c$ for all $c \in [a, b] \cap \mathbb{R}$, then G has a critical point $u \neq 0$ with either*

$$a < G(u) < 0, \quad C_{d-1}(G, u) \neq 0$$

or

$$0 < G(u) < b, \quad C_{d+1}(G, u) \neq 0.$$

When G is bounded from below, taking $a < \inf G(E)$ and $b = +\infty$ gives the following three critical points theorem; see also Krasnosel'skii [62], Chang [28], Liu and Li [83], and Liu [82].

Corollary 1.1.16 *If G has a local linking near the origin with $\dim N = d \geqslant 2$, is bounded from below, has only a finite number of critical points, and satisfies (C), then G has a global minimizer $u_0 \neq 0$ with*

$$G(u_0) < 0, \quad C_q(G, u_0) = \delta_{q0}\, \mathcal{G}$$

and a critical point $u \neq 0, u_0$ with either

$$G(u) < 0, \quad C_{d-1}(G, u) \neq 0$$

or

$$G(u) > 0, \quad C_{d+1}(G, u) \neq 0.$$

Proposition 1.1.14, Theorem 1.1.15, and Corollary 1.1.16 will be proved under a generalized notion of local linking in Section 1.9; see also Perera [107].

1.2 Compactness conditions

It is usually necessary to have some "compactness" when seeking critical points of a functional. The following condition was originally introduced by Palais and Smale [101].

Definition 1.2.1 G satisfies the Palais–Smale compactness condition at the level c, or $(\mathrm{PS})_c$ for short, if every sequence $(u_j) \subset E$ such that

$$G(u_j) \to c, \quad G'(u_j) \to 0,$$

called a $(\mathrm{PS})_c$ sequence, has a convergent subsequence; G satisfies (PS) if it satisfies $(\mathrm{PS})_c$ for every $c \in \mathbb{R}$, or equivalently, if every sequence such that

$$(G(u_j)) \text{ is bounded}, \quad G'(u_j) \to 0,$$

called a (PS) sequence, has a convergent subsequence.

The following weaker version was introduced by Cerami [25].

Definition 1.2.2 G satisfies the Cerami condition at the level c, or $(\mathrm{C})_c$ for short, if every sequence such that

$$G(u_j) \to c, \quad \big(1 + \|u_j\|\big)\, G'(u_j) \to 0,$$

called a $(\mathrm{C})_c$ sequence, has a convergent subsequence; G satisfies (C) if it satisfies $(\mathrm{C})_c$ for every c, or equivalently, if every sequence such that

$$(G(u_j)) \text{ is bounded}, \quad \big(1 + \|u_j\|\big)\, G'(u_j) \to 0,$$

called a (C) sequence, has a convergent subsequence.

This condition is weaker since a $(\mathrm{C})_c$ (resp. (C)) sequence is clearly a $(\mathrm{PS})_c$ (resp. (PS)) sequence also. Note that the limit of a $(\mathrm{PS})_c$ (resp. (PS)) sequence is in K^c (resp. K) since G is C^1. Since any sequence in K^c is a $(\mathrm{C})_c$ sequence, it follows that K^c is compact when $(\mathrm{C})_c$ holds.

1.3 Deformation lemmas

Deformation lemmas allow us to lower sublevel sets of a functional, away from its critical set, and are an essential tool for locating critical points. The main ingredient in their proofs is usually a suitable negative pseudo-gradient flow, a notion due to Palais [103].

Definition 1.3.1 A pseudo-gradient vector field for G on $\widehat{E}$ is a locally Lipschitz continuous mapping $V : \widehat{E} \to E$ satisfying

$$\|V(u)\| \leqslant \left\|G'(u)\right\|, \quad 2\left(G'(u), V(u)\right) \geqslant \left\|G'(u)\right\|^2 \quad \forall u \in \widehat{E}. \tag{1.1}$$

Lemma 1.3.2 *There is a pseudo-gradient vector field V for G on $\widehat{E}$. When G is even, V may be chosen to be odd.*

Proof For each $u \in \widehat{E}$, there is a $w(u) \in E$ satisfying

$$\|w(u)\| < \left\|G'(u)\right\|, \quad 2\left(G'(u), w(u)\right) > \left(\left\|G'(u)\right\|\right)^2$$

by the definition of the norm in E'. Since G' is continuous, then

$$\|w(u)\| \leqslant \left\|G'(v)\right\|, \quad 2\left(G'(v), w(u)\right) \geqslant \left(\left\|G'(v)\right\|\right)^2 \quad \forall v \in N_u \tag{1.2}$$

for some open neighborhood $N_u \subset \widehat{E}$ of u.

Since $\widehat{E}$ is a metric space and hence paracompact, the open covering $\{N_u\}_{u \in \widehat{E}}$ has a locally finite refinement, i.e. an open covering $\{N_\lambda\}_{\lambda \in \Lambda}$ of $\widehat{E}$ such that

(*i*) each $N_\lambda \subset N_{u_\lambda}$ for some $u_\lambda \in \widehat{E}$,
(*ii*) each $u \in \widehat{E}$ has a neighborhood U_u that intersects N_λ only for λ in some finite subset Λ_u of Λ

(see, e.g., Kelley [61]). Let $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ be a Lipschitz continuous partition of unity subordinate to $\{N_\lambda\}_{\lambda \in \Lambda}$, i.e.

(*i*) $\varphi_\lambda \in \operatorname{Lip}\left(\widehat{E}, [0, 1]\right)$ vanishes outside N_λ,
(*ii*) for each $u \in \widehat{E}$,

$$\sum_{\lambda \in \Lambda} \varphi_\lambda(u) = 1, \tag{1.3}$$

where the sum is actually over a subset of Λ_u,

for example,

$$\varphi_\lambda(u) = \frac{\operatorname{dist}(u, \widehat{E} \backslash N_\lambda)}{\sum_{\lambda \in \Lambda} \operatorname{dist}(u, \widehat{E} \backslash N_\lambda)}.$$

Now

$$V(u) = \sum_{\lambda \in \Lambda} \varphi_\lambda(u)\, w(u_\lambda)$$

is Lipschitz in each U_u and satisfies (1.1) by (1.2) and (1.3).