# 1

# Definition of $\zeta(s)$, $Z(t)$ and basic notions

## 1.1 The basic notions

The classical Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1} \qquad (s = \sigma + it,\ \sigma > 1) \tag{1.1}$$

admits analytic continuation to $\mathbb{C}$. It is regular on $\mathbb{C}$ except for a simple pole at $s = 1$. The product representation in (1.1) shows that $\zeta(s)$ does not vanish for $\sigma > 1$. The Laurent expansion of $\zeta(s)$ at $s = 1$ reads

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + \gamma_2(s-1)^2 + \cdots,$$

where the so-called *Stieltjes constants* $\gamma_k$ are given by

$$\gamma_k = \frac{(-1)^k}{k!} \lim_{N\to\infty} \left( \sum_{m \le N} \frac{\log^k m}{m} - \frac{\log^{k+1} N}{k+1} \right) \qquad (k = 0, 1, 2, \ldots).$$

In particular

$$\gamma \equiv \gamma_0 = \lim_{N\to\infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \log N \right) = -\Gamma'(1) = 0.577\,2157\ldots$$

is the *Euler constant* and

$$\Gamma(s) = \int_0^{\infty} x^{s-1} \mathrm{e}^{-x}\, \mathrm{d}x \qquad (\Re\mathrm{e}\, s > 0) \tag{1.2}$$

is the familiar *Euler gamma-function*.

The product in (1.1) is called the *Euler product*. As usual, $p$ denotes prime numbers, so that by its very essence $\zeta(s)$ represents an important tool for the investigation of prime numbers. This is even more evident from the relation

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s} \qquad (\sigma > 1),$$

which follows by logarithmic differentiation of (1.1), where *the von Mangoldt function* $\Lambda(n)$ is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha, \\ 0 & \text{if } n \neq p^\alpha \quad (\alpha \in \mathbb{N}). \end{cases}$$

The zeta-function can be also used to generate many other important arithmetic functions; for example,

$$\frac{\zeta(s)}{\zeta(2s)} \quad (\sigma > 1), \qquad \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} \quad (\sigma > \tfrac{1}{2})$$

generate the characteristic functions of *squarefree* and *squarefull* numbers, respectively. One also has, for a given $k \in \mathbb{N}$,

$$\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n)n^{-s} \qquad (\sigma > 1), \tag{1.3}$$

where the (general) divisor function $d_k(n)$ represents the number of ways $n$ can be written as a product of $k$ factors, so that in particular $d_1(n) \equiv 1$ and $d_2(n) = \sum_{\delta|n} 1$ is the number of positive divisors of $n$. The function $d_k(n)$ is a multiplicative function of $n$ (meaning $d_k(mn) = d_k(m)d_k(n)$ if $m$ and $n$ are coprime), and

$$d_k(p^\alpha) = (-1)^\alpha \binom{-k}{\alpha} = \frac{k(k+1)\cdots(k+\alpha-1)}{\alpha!}$$

for primes $p$ and $\alpha \in \mathbb{N}$.

Another significant aspect of $\zeta(s)$ is that it can be generalized to many other similar Dirichlet series (notably to the Selberg class $\mathcal{S}$, which will be discussed in Chapter 3). A vast body of literature exists on many facets of zeta-function theory, such as the distribution of its zeros and power moments of $|\zeta(\frac{1}{2}+it)|$ (see, e.g., the monographs [Iv1], [Iv4], [Mot1], [Ram] and [Tit3]). It is within this framework that the classical Hardy function (see, e.g., [Iv1]) $Z(t)$ $(t \in \mathbb{R})$ arises, and plays an important rôle in the theory of $\zeta(s)$. It is defined as

$$Z(t) := \zeta(\tfrac{1}{2}+it)\big(\chi(\tfrac{1}{2}+it)\big)^{-1/2}, \tag{1.4}$$

where $\chi(s)$ comes from the well-known functional equation for $\zeta(s)$; see (1.5) and (1.6) below. The basic properties of $Z(t)$ will be discussed in Section 1.3.

## 1.2 The functional equation for $\zeta(s)$

The functional equation is one of the most fundamental tools of zeta-function theory. Therefore we shall, for the sake of completeness, provide a proof which incidentally originated with the great German mathematician B. Riemann (1826-1866), who founded the theory of $\zeta(s)$ in his epoch-making memoir [Rie].

**Theorem 1.1** *The function $\zeta(s)$ admits analytic continuation to $\mathbb{C}$, where it satisfies the functional equation*

$$\pi^{-s/2}\zeta(s)\Gamma(\tfrac{1}{2}s) = \pi^{-(1-s)/2}\zeta(1-s)\Gamma(\tfrac{1}{2}(1-s)). \tag{1.5}$$

**Remark 1.2** The functional equation (1.5) is in a symmetric form. Alternatively we can write (1.5) as

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{1.6}$$

where we set

$$\chi(s) = \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)}\pi^{s-1/2}.$$

This expression can be put into other equivalent forms. For example, we have

$$\chi(s) = 2^s\pi^{s-1}\sin(\tfrac{1}{2}\pi s)\Gamma(1-s) = \frac{(2\pi)^s}{2\Gamma(s)\cos(\pi s/2)}, \tag{1.7}$$

where we used the well-known identities

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \quad \Gamma(s)\Gamma(s+\tfrac{1}{2}) = 2^{1-2s}\sqrt{\pi}\,\Gamma(2s). \tag{1.8}$$

**Remark 1.3** Note that (1.6) gives the identity

$$\chi(s)\chi(1-s) = 1. \tag{1.9}$$

All identities (1.5)-(1.9) hold for $s \in \mathbb{C}$.

Before we proceed to the proof of the functional equation (1.5), we need a result on a transformation formula for *the theta-function* (see (1.14)), embodied in the following lemma.

**Lemma 1.4** *We have*

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2 t} = \frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2/t} \qquad (t > 0). \tag{1.10}$$

*Proof of Lemma 1.4* For $v \in \mathbb{R}$, $\tau = iy$, $y > 0$ we have the Fourier expansion

$$f(v) := \sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi i\tau(n+v)^2} = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{2\pi ikv}, \tag{1.11}$$

since $f(v)$ is periodic with period 1 and $f(v) \in C^1[0, 1]$. Hence with $A = -2\pi ik$, $B = \pi y$, we have for the Fourier coefficients $c_k$ the expression

$$\begin{aligned} c_k &= \int_0^1 \sum_{n=-\infty}^{\infty} \mathrm{e}^{\pi i\tau(n+v)^2 - 2\pi ikv}\,\mathrm{d}v \\ &= \sum_{n=-\infty}^{\infty} \int_0^1 \mathrm{e}^{\pi i\tau(n+v)^2 - 2\pi ik(n+v)}\,\mathrm{d}v \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-\pi yv^2 - 2\pi ikv}\,\mathrm{d}v = \int_{-\infty}^{\infty} \mathrm{e}^{Av - Bv^2}\,\mathrm{d}v \\ &= \sqrt{\frac{\pi}{B}}\mathrm{e}^{A^2/(4B)} = \frac{1}{\sqrt{y}}\mathrm{e}^{-\pi k^2/y}. \end{aligned} \tag{1.12}$$

The change of order of summation and integration in (1.12) is justified by absolute convergence. Here we used the classical integral

$$\int_{-\infty}^{\infty} \exp(At - Bt^2)\,\mathrm{d}t = \sqrt{\frac{\pi}{B}} \exp\left(\frac{A^2}{4B}\right) \qquad (\Re\mathrm{e}\, B > 0). \tag{1.13}$$

Setting $v = 0$, $i\tau = i^2 y = -t$, $y = t$ in (1.11) and (1.12), we obtain (1.10) of Lemma 1.4. By analytic continuation it is seen that (1.10) remains valid for $\Re\mathrm{e}\, t > 0$. If we define the theta-function as

$$\vartheta(t) = \sum_{n=1}^{\infty} \mathrm{e}^{-\pi n^2 t} \qquad (\Re\mathrm{e}\, t > 0), \tag{1.14}$$

then (1.10) yields the transformation formula

$$\vartheta(t) = \frac{1}{2\sqrt{t}}\left(2\vartheta\left(\frac{1}{t}\right) + 1\right) - \frac{1}{2} \qquad (\Re\mathrm{e}\, t > 0). \tag{1.15}$$

*Proof of Theorem 1.1* We start from

$$\Gamma(\tfrac{1}{2}s) = \int_0^{\infty} \mathrm{e}^{-x} x^{s/2-1}\,\mathrm{d}x \qquad (\sigma > 0),$$

which is just (1.2) with $s/2$ in place of $s$. If $n \in \mathbb{N}$, we write $\pi n^2 x$ in place of $x$ to obtain

$$\Gamma(\tfrac{1}{2}s) = \pi^{s/2} n^s \int_0^{\infty} \mathrm{e}^{-\pi n^2 x} x^{s/2-1}\,\mathrm{d}x \qquad (\sigma > 0),$$

or

$$n^{-s} = \frac{\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty e^{-\pi n^2 x} x^{s/2-1}\, dx \qquad (\sigma > 0).$$

Summation over $n$ gives, for $\sigma > 1$,

$$\zeta(s) = \sum_{n=1}^\infty n^{-s} = \frac{\pi^{s/2}}{\Gamma(s/2)} \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 x} x^{s/2-1}\, dx.$$

Since the series

$$\sum_{n=1}^\infty \int_0^\infty \left| e^{-\pi n^2 x} x^{s/2-1} \right| dx = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 x} x^{\sigma/2-1}\, dx = \sum_{n=1}^\infty \Gamma(\sigma/2)\pi^{-\sigma/2} n^{-\sigma}$$

converges for $\sigma > 1$, we can change the order of summation and integration to obtain

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty \vartheta(x) x^{s/2-1}\, dx. \tag{1.16}$$

In view of (1.15) we may write (1.16) as

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \int_0^1 x^{s/2-1}\vartheta(x)\, dx + \int_1^\infty x^{s/2-1}\vartheta(x)\, dx \\
&= \int_0^1 x^{s/2-1}\Big(x^{-1/2}\vartheta\Big(\frac{1}{x}\Big) + \frac{1}{2}x^{-1/2} - \frac{1}{2}\Big)\, dx \\
&\quad + \int_1^\infty x^{s/2-1}\vartheta(x)\, dx \\
&= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 x^{s/2-3/2}\vartheta(1/x)\, dx + \int_1^\infty x^{s/2-1}\vartheta(x)\, dx \\
&= \frac{1}{s(s-1)} + \int_1^\infty \Big(x^{-s/2-1/2} + x^{s/2-1}\Big)\vartheta(x)\, dx. \qquad (1.17)
\end{aligned}$$

Note first that the last expression in (1.17) remains invariant if $s$ is replaced by $1-s$. Secondly, the last integral in (1.17) converges uniformly (since $x \ge 1$ in the integrand) in any strip

$$-\infty < a \le \sigma = \mathbb{R}\mathrm{e}\, s \le b < +\infty.$$

Consequently the last integral in (1.17) represents an entire function of $s$. Therefore

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) - \frac{1}{s(s-1)}$$

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is an entire function of $s$. Since $\pi^{s/2}/\Gamma(s/2)$ is an entire function (because $\Gamma(s)$ has no zeros),

$$\zeta(s) - \frac{1}{s(s-1)}\frac{\pi^{s/2}}{2\Gamma(s/2)}$$

is also an entire function. Further, since $s\Gamma(s/2) = 2\Gamma(s/2+1)$, it follows that

$$\zeta(s) - \frac{1}{s-1}\frac{\pi^{s/2}}{2\Gamma(s/2+1)}$$

is an entire function. Since $\sqrt{\pi}/(2\Gamma(3/2)) = 1$, we have that $\zeta(s) - 1/(s-1)$ is an entire function, thus $\zeta(s)$ is regular in $\mathbb{C}$ except for a simple pole at $s = 1$ with residue 1.

This discussion shows that (1.17) provides analytic continuation of $\zeta(s)$ to $\mathbb{C}$, as well as the functional equation (1.5).

**Corollary 1.5** *If we define*

$$\eta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad \xi(s) = \tfrac{1}{2}s(s-1)\eta(s), \tag{1.18}$$

*then $\xi(s)$ is an entire function of $s$ satisfying the functional equation $\xi(s) = \xi(1-s)$. It is real for $t = 0$ and $\sigma = 1/2$ and $\xi(0) = \xi(1) = 1/2$.*

## 1.3 Properties of Hardy's function

We continue with the discussion of $Z(t)$. Recall that the zeta, sine and the gamma-function take conjugate values at conjugate points. Hence it follows from (1.7) and (1.9) that

$$\overline{\chi(\tfrac{1}{2}+it)} = \chi(\tfrac{1}{2}-it) = \chi^{-1}(\tfrac{1}{2}+it),$$

so that (1.4) gives $Z(t) \in \mathbb{R}$ when $t \in \mathbb{R}$, and $|Z(t)| = |\zeta(\frac{1}{2}+it)|$. Thus the zeros of $\zeta(s)$ on the "critical line" $\Re e\, s = 1/2$ are in one-to-one correspondence with the real zeros of $Z(t)$. This property makes $Z(t)$ an invaluable tool in the study of the zeros of the zeta-function on the critical line. If we use (1.5) and (1.6) we have

$$\left(\chi(\tfrac{1}{2}+it)\right)^{-1/2} = \pi^{-it/2}\frac{\Gamma^{1/2}(\frac{1}{4}+\frac{1}{2}it)}{\Gamma^{1/2}(\frac{1}{4}-\frac{1}{2}it)} = \pi^{-it/2}\frac{\Gamma(\frac{1}{4}+\frac{1}{2}it)}{|\Gamma(\frac{1}{4}+\frac{1}{2}it)|} := \mathrm{e}^{i\theta(t)},$$

say, where $\theta(t)$ is a smooth function for which

$$\theta(t) = -\frac{1}{2i}\log\chi(\tfrac{1}{2}+it), \qquad \theta'(t) = -\frac{1}{2}\frac{\chi'(\frac{1}{2}+it)}{\chi(\frac{1}{2}+it)}.$$

Note that also

$$\theta(t) = \mathbb{I}\mathrm{m}\,\left\{\log\Gamma(\tfrac14 + \tfrac12 it)\right\} - \tfrac12 t\log\pi \quad \in \mathbb{R}\mathrm{e} \tag{1.19}$$

if $t \in \mathbb{R}$, thus $\theta(0) = 0$. The function $\theta(t)$ is odd, since in view of $\chi(s)\chi(1-s) = 1$ we have

$$\begin{aligned}\theta(-t) &= -\frac{1}{2i}\log\chi(\tfrac12 - it) = -\frac{1}{2i}\log\frac{1}{\chi(\frac12 + it)}\\ &= \frac{1}{2i}\log\chi(\tfrac12 + it) = -\theta(t).\end{aligned}$$

It is also monotonic increasing for $t \geq 7$, which follows from formulas (1.21)-(1.22) below. We may write $Z(t)$ alternatively as

$$Z(t) = \mathrm{e}^{i\theta(t)}\zeta(\tfrac12 + it), \quad \mathrm{e}^{i\theta(t)} := \pi^{-it/2}\frac{\Gamma(\frac14 + \frac12 it)}{|\Gamma(\frac14 + \frac12 it)|} \quad (\theta(t) \in \mathbb{R}). \tag{1.20}$$

It is also useful to note that $Z(t)$ is an even function of $t$, because

$$\begin{aligned}Z(-t) &= \zeta(\tfrac12 - it)\Big(\chi(\tfrac12 - it)\Big)^{-1/2} = \zeta(\tfrac12 + it)\Big(\chi(\tfrac12 - it)\Big)^{1/2}\\ &= \zeta(\tfrac12 + it)\big(\chi(\tfrac12 + it)\big)^{-1/2} = Z(t).\end{aligned}$$

We have the explicit representation

$$\theta(t) = \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \Delta(t). \tag{1.21}$$

Here (see Lemma 5.1 for a proof; here $\Delta(t)$ is not to be confused with the error term in the Dirichlet divisor problem)

$$\Delta(t) := \frac{t}{4}\log\left(1 + \frac{1}{4t^2}\right) + \frac14\arctan\frac{1}{2t} + \frac{t}{2}\int_0^\infty \frac{\psi(u)}{(u+\frac14)^2 + (\frac{t}{2})^2}\,\mathrm{d}u \tag{1.22}$$

with

$$\psi(x) = x - [x] - \tfrac12 = -\sum_{n=1}^\infty \frac{\sin(2n\pi x)}{n\pi} \qquad (x \notin \mathbb{Z}).$$

The representation (1.21)-(1.22) follows from *Stirling's formula* (see (2.14)) for the gamma-function in the form

$$\log\Gamma(s) = (s - 1/2)\log s - s + \log\sqrt{2\pi} - \int_0^\infty \frac{\psi(u)}{u+s}\,\mathrm{d}u,$$

which in turn is a consequence of the product formula

$$\frac{1}{\Gamma(s)} = s\exp(\gamma s)\prod_{n=1}^{\infty}\Big(1+\frac{s}{n}\Big)\,\mathrm{e}^{-s/n}. \tag{1.23}$$

Note that (1.23) valid for $s \in \mathbb{C}$, and can serve as a definition of $\Gamma(s)$ equivalent to (1.2).

The expression (1.21) is very useful, since it allows one to evaluate explicitly all the derivatives of $\theta(t)$. For $t \to \infty$ it is seen that $\Delta(t)$ admits an asymptotic expansion in terms of negative powers of $t$, and from (1.19) and Stirling's formula it is found that ($B_k$ is the $k$th Bernoulli number)

$$\Delta(t) \ \sim \ \sum_{n=1}^{\infty}\frac{(2^{2n}-1)|B_{2n}|}{2^{2n}(2n-1)2nt^{2n-1}}. \tag{1.24}$$

The meaning of $\sim$ in (1.24) is that, for an arbitrary integer $N \ge 1$, $\Delta(t)$ equals the sum of the first $N$ terms of the series in (1.24), plus the error term, which is $O_N(t^{-2N-1})$. In general we shall have, for $k \ge 0$ and suitable constants $c_{k,n}$,

$$\Delta^{(k)}(t) \ \sim \ \sum_{n=1}^{\infty} c_{k,n}t^{1-2n-k}. \tag{1.25}$$

Thus (1.21) and (1.24) give

$$\theta(t) \ \sim \ \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \sum_{n=1}^{\infty}\frac{(2^{2n-1}-1)|B_{2n}|}{2^{2n}(2n-1)2nt^{2n-1}}, \tag{1.26}$$

and we also have asymptotic expansions for the derivatives of $\theta(t)$. In particular, we have the approximations

$$\begin{aligned}\theta(t) &= \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + O\Big(\frac{1}{t^5}\Big),\\ \theta'(t) &= \frac{1}{2}\log\frac{t}{2\pi} + O\Big(\frac{1}{t^2}\Big),\\ \theta''(t) &= \frac{1}{2t} + O\Big(\frac{1}{t^3}\Big),\end{aligned} \tag{1.27}$$

which are sufficiently sharp for many applications.

## 1.4 The distribution of zeta-zeros

In what concerns the distribution of zeros of $\zeta(s)$, it is known that $\zeta(s)$ has no zeros in the region

$$\sigma \ge 1 - C(\log t)^{-2/3}(\log\log t)^{-1/3} \qquad (C > 0,\ t \ge t_0 > 0). \tag{1.28}$$

This result, the strongest so-called *zero-free region* for $\zeta(s)$ even today, was obtained by an application of I. M. Vinogradov's method of exponential sums. In a modern form, the crucial bound which implies (1.28) states (see, e.g., [Iv1, chapter 6]) that

$$\sum_{N<n\le N_1\le 2N} n^{it} \ll N\exp\Big(-\frac{C\log^3 N}{\log^2 t}\Big) \qquad (C>0) \tag{1.29}$$

for $N_0 \le N \le \frac{1}{2}t,\ t \ge t_0$. From (1.5) it follows that $\zeta(-2n) = 0$ for $n \in \mathbb{N}$. These zeros are the only real zeros of $\zeta(s)$, and are called the *trivial zeros* of $\zeta(s)$. In 1859, B. Riemann [Rie] calculated a few complex zeros of $\zeta(s)$ and found that they lie on the line $\mathbb{R}\mathrm{e}\, s = \frac{1}{2}$, which is called the *critical line* in the theory of $\zeta(s)$. The first ten pairs of complex zeros (arranged in size according to their absolute value) are (see, e.g., C. B. Haselgrove [Has])

$$\begin{aligned}
&\frac12 \pm i14.134\,725\ldots, \quad \frac12 \pm i21.022\,039\ldots, \quad \frac12 \pm i25.010\,857\ldots,\\
&\frac12 \pm i30.424\,876\ldots, \quad \frac12 \pm i32.935\,061\ldots, \quad \frac12 \pm i37.586\,178\ldots,\\
&\frac12 \pm i40.918\,719\ldots, \quad \frac12 \pm i43.327\,073\ldots, \quad \frac12 \pm i48.005\,150\ldots,\\
&\frac12 \pm i49.773\,832\ldots.
\end{aligned}$$

The number of complex zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \le T$ (multiplicities included) is denoted by $N(T)$. The asymptotic formula for $N(T)$ is the famous *Riemann-von Mangoldt formula*. It was enunciated by B. Riemann [Rie] in 1859, but proved by H. von Mangoldt [Man] in 1895. We state it here as follows.

**Theorem 1.6** *Let*

$$S(T) := \frac{1}{\pi}\arg\zeta(\tfrac12 + iT). \tag{1.30}$$

*Then*

$$N(T) = \frac{T}{2\pi}\log\Big(\frac{T}{2\pi}\Big) - \frac{T}{2\pi} + \frac78 + S(T) + O\Big(\frac1T\Big), \tag{1.31}$$

*where the $O$-term is a continuous function of $T$, and*

$$S(T) \;=\; O(\log T). \tag{1.32}$$

Here $\arg\zeta(\frac12 + iT)$ is evaluated by continuous variation starting from $\arg\zeta(2) = 0$ and proceeding along straight lines, first up to $2 + iT$ and then to

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$1/2 + iT$, assuming that $T$ is not an ordinate of a zeta zero. If $T$ is an ordinate of a zero, then we set $S(T) = S(T+0)$.

**Remark 1.7** On the RH (the Riemann hypothesis, that all complex zeros of $\zeta(s)$ have real parts equal to 1/2) one can slightly improve (1.32) and obtain that (see [Tit3])

$$S(T) \;=\; O\left(\frac{\log T}{\log\log T}\right). \tag{1.33}$$

*Proof of Theorem 1.6* Let $\mathcal{D}$ be the rectangle with vertices $2 \pm iT$, $-1 \pm iT$, where $T$ ($>3$) is not an ordinate of a zero. The function $\xi(s)$, defined by (1.18), has $2N(T)$ zeros in the interior of $\mathcal{D}$, and none on the boundary. Therefore we have

$$N(T) \;=\; \frac{1}{4\pi}\mathbb{I}\mathrm{m}\left(\int_{\mathcal{D}} \frac{\xi'(s)}{\xi(s)}\,\mathrm{d}s\right). \tag{1.34}$$

Logarithmic differentiation of (1.18) gives

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} + \frac{\eta'(s)}{\eta(s)},$$

where $\eta(s)$ is also given by (1.18). Observe first that

$$\mathbb{I}\mathrm{m}\left\{\int_{\mathcal{D}}\left(\frac{1}{s} + \frac{1}{s-1}\right)\mathrm{d}s\right\} = 4\pi.$$

Next, note that $\eta(s) = \eta(1-s)$ and $\eta(\sigma \pm it)$ are conjugates, so that

$$\int_{\mathcal{D}}\left(\frac{\eta'(s)}{\eta(s)}\,\mathrm{d}s\right) = 4\mathbb{I}\mathrm{m}\int_{\mathcal{L}}\left(\frac{\eta'(s)}{\eta(s)}\,\mathrm{d}s\right),$$

where $\mathcal{L}$ consists of the segments $[2,\ 2+iT]$ and $[2+iT,\ 1/2+iT]$. Therefore

$$\begin{aligned}\mathbb{I}\mathrm{m}\int_{\mathcal{L}}\left(\frac{\eta'(s)}{\eta(s)}\,\mathrm{d}s\right) &= \mathbb{I}\mathrm{m}\left\{\int_{\mathcal{L}}\left(-\frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)}\right)\mathrm{d}s\right\}\\ &= -\frac{1}{2}(\log\pi)T + \mathbb{I}\mathrm{m}\left(\int_{\mathcal{L}}\frac{\Gamma'(s/2)}{2\Gamma(s/2)}\,\mathrm{d}s + \int_{\mathcal{L}}\frac{\zeta'(s)}{\zeta(s)}\,\mathrm{d}s\right).\end{aligned}$$

Note that

$$\mathbb{I}\mathrm{m}\left(\int_{\mathcal{L}}\frac{\Gamma'(s/2)}{2\Gamma(s/2)}\,\mathrm{d}s\right) \;=\; \mathbb{I}\mathrm{m}\,\log\Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}iT\right), \tag{1.35}$$

and using Stirling's formula in the form (2.16) we have

$$\mathbb{I}\mathrm{m}\left(\int_{\mathcal{L}}\frac{\Gamma'(s/2)}{2\Gamma(s/2)}\,\mathrm{d}s\right) = \frac{1}{2}T\log\left(\frac{T}{2}\right) - \frac{T}{2} - \frac{\pi}{8} + O\left(\frac{1}{T}\right),$$