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978-1-107-02823-4 — Introduction to the Network Approximation Method for Materials Modeling
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Excerpt
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Review of mathematical notions used in the analysis of transport problems in densely-packed composite materials

In this chapter, for the convenience of the reader, we briefly recall several basic notions used in the network approximation method which will be used throughout the book.

The network approximation method intensively addresses three branches of mathematics: graph theory, the theory of partial differential equations and duality theory (convex analysis). These branches are very wide and we present here the minimal necessary information. Detailed explanations can be found in the literature referred to in the corresponding sections of our review.

1.1 Graphs

In this section, we present basic notions related to networks. The study of networks, in the form of mathematical graph theory (see, e.g., West (2000)) is the fundamental cornerstone of discrete mathematics.

1.1.1 General information about graphs (networks)

A network is a set of items, which we call vertices or nodes, with connections between them, called edges, see Figure 1.1. In the mathematical literature networks are often called graphs.

Two vertices $\mathbf{x}_i$ and $\mathbf{x}_j$ connected by an edge e_{ij} are called adjacent. The connections in a network can be represented by the collection of the edges $\{e_{ij}\}$ or by the connectivity matrix G_{ij} determined as

$$G_{ij} = \begin{cases} 1 & \text{if the } i\text{-th and } j\text{-th vertices are connected,} \\ 0 & \text{otherwise; } i, j = 1, \dots, N. \end{cases} \tag{1.1.1}$$

A graph can be described by the set $\mathbf{X} = \{\mathbf{x}_i;\ i = 1, \dots, N\}$ of its vertices and the set $\mathbf{E} = \{e_{ij};\ i, j = 1, \dots, N\}$ of its edges or by the set $\mathbf{X}$ of vertices and the

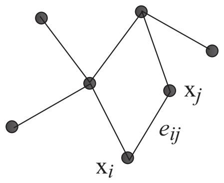

Figure 1.1 A graph.

connectivity matrix $\mathbf{G} = \{G_{ij};\ i, j = 1, \ldots, N\}$. Thus, a graph can be represented in the form $\mathcal{G} = (\mathbf{X}, \mathbf{E})$ or $\mathcal{G} = (\mathbf{X}, \mathbf{G})$.

Both vertices and edges may have a variety of properties associated with them. In many cases some numbers (called the weights of the edges) are assigned in correspondence to the edges. A graph with weights is called a weighted graph. For a weighted graph, the definition (1.1.1) is modified as follows

$$G_{ij} = \begin{cases} g_{ij} \neq 0 & \text{if the } i\text{-th and } j\text{-th vertices are connected,} \\ 0 & \text{otherwise; } i, j = 1, \ldots, N. \end{cases} \tag{1.1.2}$$

The weights g_{ij} (1.1.2) include complete information about the connectivity matrix G_{ij} (1.1.1).

The edges of a graph can also be directed. Graphs composed of directed edges are called directed graphs. Directed weighted graphs are described by (1.1.2) with the weights taking positive or negative values. In specific problems the weights g_{ij} will represent fluxes in networks such as fluxes of fluid, electric current, heat, etc. A weighted graph can be described by the set $(\mathbf{X}, \mathbf{G}) = \{x_i,\ g_{ij};\ i, j = 1, \ldots, N\}$.

The size of the network is defined as the total number N of vertices in the network. In this book, we consider networks of finite (possibly large) size.

We say that a network (graph) is connected if every two vertices of the network are connected by a path consisting of edges in this network. A loop in the network is a path, which begins and ends at the same vertex.

1.1.2 Delaunay–Voronoi graphs in modeling of disordered structures

The key condition for structural modeling of disordered particle-filled composites is the high intensity of physical fields in the gaps between closely spaced neighboring particles. That is why the notion of neighboring particles plays an important role. While for periodic arrays of particles the notion of neighbors is obvious, for disordered (non-periodic) arrays the formal definition of neighbors requires an effort.

One way of introducing such a notion is by employing the Delaunay–Voronoi method (Aurenhammer and Klein, 2000; Sahimi, 2003). We now briefly outline this method. Recall that the Voronoi tessellation (also known as Dirichlet tessellations and sometimes referred to as Wigner–Seiz tessellations (Sahimi, 2003) of a

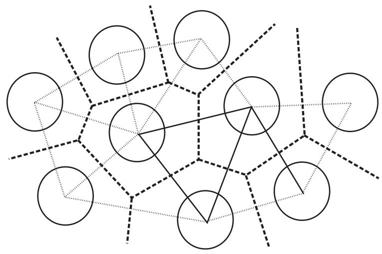

Figure 1.2 Voronoi polygons (tessellations) and Delaunay graph in dimension two.

collection of geometric objects, called the generating points (e.g., a collection of centers of particles in our case), is a partition of the space into Voronoi cells, each of which consists of the points of the space, which are closer to one particular object than to any others.

It is well known that for a given set of generating points in a plane (space) the Voronoi cells are polygons (polyhedra). The generating points which share a common edge (face) in such a partition are called *neighbors*.

While the faces (edges) of Voronoi cells in two dimensions form a graph, in dimension three, this tessellation does not define a graph. However, in all dimensions a graph, called a *Delaunay graph*, can be introduced based on a given Voronoi tessellation. The edges of the Delaunay graph are obtained by connecting neighboring generating points in the Voronoi tessellation. Sometimes the Delaunay graph is referred to as a Voronoi graph (Sahimi, 2003).

Finally, we remark that the Delaunay graph is connected; that is, every two vertices of this graph are connected by a path, which consists of edges of this graph, see Figure 1.2.

1.2 Functional spaces and weak solutions of partial differential equations

In this section we briefly recall several basic notions from the theory of partial differential equations. Detailed explanations can be found in (Kato, 1976; Kolmogorov and Fomin, 1970; Lions and Magenes, 1972; Mizohata, 1973; Schwartz, 1966; Yosida, 1971).

1.2.1 Distributions and distributional derivatives

A complete normed linear space is called a Banach space. A Banach space H with the norm defined as $||x||_H = \sqrt{(x, x)_H}$, where $(x, y)_H$ means the scalar product in H, is called a Hilbert space.

Denote by $\mathcal{D}(Q)$ the set of infinitely differentiable functions with compact support (a function has compact support if it is equal to zero outside a compact set $K \subset Q$), where Q is a bounded domain in $\mathbb{R}^N$.

Topology (convergence) on $\mathcal{D}(Q)$ is introduced as follows: let $\Theta^i \in \mathcal{D}(Q)$, $i = 1, 2, \ldots$; $\Theta^0 \in \mathcal{D}(Q)$. Then the convergence $\Theta^i \to \Theta^0$ in $\mathcal{D}(Q)$ as $i \to \infty$ means that the supports of all functions belong to the same compact set from Q, and Θ^i and all its derivatives converge uniformly to Θ^0 and its corresponding derivatives.

Now, let T be a linear continuous functional determined on $\mathcal{D}(Q)$, i.e., a map (a rule) that assigns to every $\Theta \in \mathcal{D}(Q)$ a number $\langle T, \Theta\rangle$, which is the value of the functional T on this function $\Theta \in \mathcal{D}(Q)$ (in other words $\langle , \rangle$ means the dual coupling). This map is linear with respect to Θ and $\langle T, \Theta^i\rangle \to \langle T, \Theta^0\rangle$ if $\Theta^i \to \Theta^0$ in $\mathcal{D}(Q)$ as $i \to \infty$. Such functionals are called distributions on Q. The set of distributions is denoted by $\mathcal{D}'(Q)$.

Every locally integrable function $f(\mathbf{x})$ on Q generates a distribution $\tilde{f} \in \mathcal{D}'(Q)$ defined as follows

$$\langle \tilde{f}, \Theta\rangle = \int_Q f(\mathbf{x})\Theta(\mathbf{x})d\mathbf{x}.$$

If $T \in \mathcal{D}'(Q)$, its distributional derivative $\dfrac{\partial T}{\partial x_i} \in \mathcal{D}'(Q)$ is determined by the equality

$$\left\langle \frac{\partial T}{\partial x_i}, \Theta \right\rangle = -\left\langle T, \frac{\partial \Theta}{\partial x_i} \right\rangle \tag{1.2.1}$$

for every $\Theta \in \mathcal{D}(Q)$.

Formula (1.2.1) of the distributional derivative reminds us of the integration by parts identity. This is not a coincidence. Historically, the notion of distributional derivative was inspired by this formula, see (Schwartz, 1966; Sobolev, 1937, 1950).

1.2.2 Sobolev functional spaces

We consider a function $f(\mathbf{x})$ defined on the domain Q such that $|f(\mathbf{x})|^p$ (the p-th power of the function) is Lebesgue integrable (Burkill, 2004; Rudin, 1964). This set of such functions is denoted by $L_p(Q)$. When supplied with the norm

$$||f||_{L_p} = \left(\int_Q |f(\mathbf{x})|^p d\mathbf{x} \right)^{1/p}$$

$L_p(Q)$ is Banach functional space.

The functional space $L_\infty(Q)$ is introduced as the set of integrable functions bounded almost everywhere (Rudin, 1964). The norm in $L_\infty(Q)$ is introduced as

$$||f||_{L_\infty} = \text{ess} \sup_{\mathbf{x}\in Q} |f(\mathbf{x})|$$

or

$$||f||_{L_\infty} = \lim_{n\to\infty} ||f||_{L_n}.$$

These two definitions are equivalent (Rudin, 1964).

For $p=2$, $L_2(Q)$ becomes a Hilbert space with the scalar product

$$(f,g)_{L_2}=\int_Q f(\mathbf{x})g(\mathbf{x})d\mathbf{x}.$$

The Sobolev functional space $W^{m,p}(Q)$ is the set of distributions which (with all its distributional derivatives of order less than or equal to m) are generated by functions from $L_p(Q)$. This is a Banach space with the norm

$$||f||_{W^{m,p}}=\left(\int_Q\sum_{0\le m_1+\ldots+m_N\le m}\left|\frac{\partial^{m_1+\ldots+m_N}f}{\partial x_1^{m_1}\ldots\partial x_N^{m_N}}\right|^p d\mathbf{x}\right)^{1/p}. \tag{1.2.2}$$

For $p=2$, $W^{m,p}(Q)$ is a Hilbert space, which is denoted by $H^m(Q)$, with scalar product

$$(f,g)_{H^m}=\int_Q\sum_{0\le m_1+\ldots+m_N\le m}\frac{\partial^{m_1+\ldots+m_N}f}{\partial x_1^{m_1}\ldots\partial x_N^{m_N}}\cdot\frac{\partial^{m_1+\ldots+m_N}g}{\partial x_1^{m_1}\ldots\partial x_N^{m_N}}d\mathbf{x}.$$

For $m=1$, the scalar product in $H^1(Q)$ is

$$(f,g)_{H^1}=\int_Q\big(f(\mathbf{x})g(\mathbf{x})+\nabla f(\mathbf{x})\nabla g(\mathbf{x})\big)d\mathbf{x}, \tag{1.2.3}$$

where

$$\nabla=\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right).$$

The norm in $H^1(Q)$ is

$$||f||_{H^1}=\sqrt{\int_Q\left(f^2(\mathbf{x})+|\nabla f(\mathbf{x})|^2\right)d\mathbf{x}}. \tag{1.2.4}$$

The functional space $H^m(Q)$ may also be introduced as the closure of the functional space $C^\infty(Q)$ in the norm (1.2.2) ($p=2$) (Adams, 1975). The closure of $\mathcal{D}(Q)$ (the space of $C^\infty(Q)$ functions with compact support) in the norm (1.2.2) or (1.2.4) is denoted by $W_0^{m,p}(Q)$ or $H_0^1(Q)$, respectively (Adams, 1975). In particular, this means that the sets $C^\infty(Q)$ and $\mathcal{D}(Q)$ are dense subsets of the respective functional spaces $H^m(Q)$ and $H_0^m(Q)$.

It is possible to introduce the space $H^s(Q)$ for real (not necessary integer or even positive) values of s. In the special case where $Q=\mathbb{R}^N$, $p=2$, the space $H^s(\mathbb{R}^N)$ can be introduced by using the Fourier transform. $H^s(\mathbb{R}^N)$ is the set of all functions $f\in L_2(\mathbb{R}^N)$ such that their Fourier transforms

$$\hat{f}(\zeta)=(2\pi)^{-\frac{N}{2}}\int_{\mathbb{R}^N}f(\mathbf{x})e^{-i(\mathbf{x},\zeta)}d\mathbf{x},\ \ \zeta=(\zeta_1,\ldots,\zeta_N)\in\mathbb{R}^N$$

satisfy the condition $|(1+|\zeta|^2)^{s/2}\hat{f}| \in L_2(\mathbb{R}^N)$. The norm of $f(\mathbf{x})$ in $H^s(\mathbb{R}^N)$ is introduced as follows

$$||f||_{H^s(\mathbb{R}^N)} = ||(1+|\zeta|^2)^{s/2}\hat{f}(\zeta)||_{L_2(\mathbb{R}^N)}.$$

This way of introducing of non-integer order derivatives is based on the well-known property of the Fourier transform of the derivative (see, e.g., Rudin (1992))

$$i\zeta_n\hat{f}(\zeta) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \frac{\partial f}{\partial x_n}(\mathbf{x})e^{i(\mathbf{x},\zeta)}d\mathbf{x}, \ n = 1, \ldots, N \tag{1.2.5}$$

and Plancherel's theorem

$$\int_{\mathbb{R}^N} f^2(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^N} \hat{f}^2(\zeta)d\zeta. \tag{1.2.6}$$

From (1.2.5) and (1.2.6) we have

$$\int_{\mathbb{R}^N} |\nabla f|^2(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^n} |\zeta|^2\hat{f}^2(\zeta)d\zeta. \tag{1.2.7}$$

For the second, third and higher order derivatives

$$(i)^2\zeta_n\zeta_m\hat{f}(\zeta) = -\zeta_n\zeta_m\hat{f}(\zeta) = (2\pi)^{-\frac{N}{2}} \int_{R^N} \frac{\partial^2 f}{\partial x_n \partial x_m}(\mathbf{x})e^{-i(\mathbf{x},\zeta)}d\mathbf{x},$$

and so on, and we conclude that the power m of the factor ζ in the expression $|\zeta|^m\hat{f}(\zeta)$ is the order of the derivatives of the original function $f(\mathbf{x})$.

While the classical derivatives are defined for integer orders only, the expression

$$\int_{\mathbb{R}^N} |\zeta|^s\hat{f}^2(\zeta)d\zeta$$

is determined for various s (integer and real, positive and negative).

In particular, from (1.2.6) and (1.2.7), it follows that the norm in $H^1(\mathbb{R}^N)$ can be defined in two equivalent ways:

$$||f||_{H^1} = \sqrt{\int_{\mathbb{R}^N} \left(f^2(\mathbf{x}) + |\nabla f(\mathbf{x})|^2\right)d\mathbf{x}} = \sqrt{\int_{\mathbb{R}^n} \left(1+|\zeta|^2\right)\hat{f}^2(\zeta)d\zeta}.$$

The functional space $H^{-s}(Q)$, $s > 0$, can be associated with the dual space of $H_0^s(Q)$ (Lions and Magenes, 1972).

1.2.3 Traces of functions from $H^m(Q)$

The material properties of particle-filled composite materials are often described by piecewise constant functions, which is why the classical formulation of the corresponding boundary-value problems does not apply. Then we will use weak formulations (see next section) in the space $H^1(Q)$. In particular, we will need to assign boundary values for the functions from $H^1(Q)$. These functions are not in general continuous and moreover they are defined almost everywhere (but not everywhere) in the domain Q. Thus assigning boundary values on ∂Q for such functions is not straightforward and requires special techniques.

Recall that the set $C^\infty(\bar{Q})$ of infinitely differentiable functions in $\bar{Q} = Q \bigcup \partial Q$ (we assume ∂Q is a C^∞-smooth surface) is dense in $H^1(Q)$ (Lions and Magenes, 1972).

Consider an $(n-1)$-dimensional surface $\Gamma \subset \partial Q$. For a function $f \in H^1(Q)$, the trace ("boundary values" of f on Γ) $f|_\Gamma : \Gamma \to \mathbb{R}$ can be defined as follows. For a given $f \in H^1(Q)$ choose $f_i \in C^\infty(Q)$, $i = 1, 2, \ldots$, such that $f_i \to f$ in $H^1(Q)$ as $i \to \infty$. For $f_i \in C^\infty(Q)$ the notion of trace is defined in the classical sense as the value of f_i on Γ. The sequence $f_i|_\Gamma$ has a limit in $H^{1/2}(\Gamma)$ (Lions and Magenes, 1972). This limit is called the trace of the function $f \in H^1(Q)$ on the surface Γ. Due to the density property the trace operator can be extended by continuity from $C^\infty(\bar{Q})$ to $H^1(Q)$ and thus it becomes a linear bounded (continuous) operator from $H^1(Q)$ to $H^{1/2}(\Gamma)$.

Similarly, the trace operator can be defined as a bounded operator from $H^m(Q)$ to $H^{m-1/2}(\Gamma)$. This definition of the trace operator allows for integration by parts for functions from Sobolev functional spaces which follows immediately from the density of $C^\infty(\bar{Q})$ in $H^m(Q)$. The space of $H^m(Q)$ functions with zero trace is $H_0^m(Q)$.

This trace theorem will often be used in the following generalized form (Lions and Magenes, 1972; Ekeland and Temam, 1976). If a domain Q has Lipschitz ($C^{0,1}$ smooth) boundary, and $\mathbf{v}$ is a three- or two-dimensional vector field, such that

$$\mathbf{v} \in L_2(Q) \text{ and } \operatorname{div} \mathbf{v} \in L_2(Q), \tag{1.2.8}$$

then the trace $\mathbf{v}(\mathbf{x}) \cdot \mathbf{n}$ is determined on $\Gamma \subseteq \partial Q$ as a function from $H^{1/2}(\Gamma)$.

A particular (but very important) case of (1.2.8) is the case of divergence-free functions from $L_2(Q)$:

$$\mathbf{v} \in L_2(Q) \text{ and } \operatorname{div} \mathbf{v} = 0. \tag{1.2.9}$$

A detailed proof of the trace theorem for divergence-free functions (1.2.9) can be found in Temam (1979).

1.2.4 Weak solutions of partial differential equations with discontinuous coefficients

An introduction to the theory of weak solutions of partial differential equations can be found in Lions and Magenes (1972). Here we present a weak formulation of an elliptic boundary-value problem.

Consider the following boundary-value problem (hereafter the summation of repeated indices is assumed if not indicated otherwise)

$$\frac{\partial}{\partial x_i}\left(a(\mathbf{x})\frac{\partial u}{\partial x_i}\right) = f(\mathbf{x}) \text{ in } Q, \tag{1.2.10}$$

$$a(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = u_1(\mathbf{x}) \text{ on } \Gamma \subseteq \partial Q, \tag{1.2.11}$$

$$u(\mathbf{x}) = u_2(\mathbf{x}) \text{ on } \partial Q \setminus \Gamma, \tag{1.2.12}$$

with coefficients

$$a(\mathbf{x}) \in L_\infty(Q) \tag{1.2.13}$$

that satisfy the coercivity condition

$$a(\mathbf{x}) \geq \gamma > 0 \text{ for all } \mathbf{x} \in Q. \tag{1.2.14}$$

If the functions $a(\mathbf{x})$, $f(\mathbf{x})$, $u_1(\mathbf{x})$, $u_2(\mathbf{x})$ and the surface ∂Q are sufficiently smooth, the problem (1.2.10) has a classical solution belonging to $C^2(Q) \bigcap C^1(\bar{C})$ (Ladyzhenskaya and Ural'tseva, 1968). Multiplying the differential equation (1.2.10) by an arbitrary function $\Theta \in C^\infty(Q)$ such that $\Theta(\mathbf{x}) = 0$ on $\partial Q \setminus \Gamma$ and integrating by parts, we obtain

$$\int_Q a(\mathbf{x})\frac{\partial u}{\partial x_i}(\mathbf{x})\frac{\partial \Theta}{\partial x_i}(\mathbf{x})d\mathbf{x} = -\int_Q f(\mathbf{x})\Theta(\mathbf{x})d\mathbf{x} + \int_\Gamma u_1(\mathbf{x})\Theta(\mathbf{x})d\mathbf{x} \tag{1.2.15}$$

for any $\Theta \in \mathcal{D}(Q)$.

The equalities (1.2.15) and (1.2.12) are defined for functions $a(\mathbf{x}) \in L_\infty(Q)$, $u_1(\mathbf{x}) \in H^{-1/2}(\partial Q)$, $u_2(\mathbf{x}) \in H^{1/2}(\partial Q)$, $u(\mathbf{x}) \in H^1(Q)$, and $\Theta(\mathbf{x}) \in H^1(Q)$.

Thus, it is possible to define the solution of the boundary-value problem (1.2.10) for non-differentiable (in particular, for piecewise continuous) coefficients $a(\mathbf{x})$. The solution of (1.2.15) is referred to as a generalized solution of the boundary-value problem (1.2.10)–(1.2.12) and it is understood in the sense of distributions.

1.2.5 Variational form of boundary value problems

Let H be a Hilbert space. A scalar function $a(u, v)$ determined on $H \times H$ is called a bilinear form on H if it is linear with respect to the first and the second variables. The bilinear form is called continuous if there exists a constant $C < \infty$ such that

$$|a(u, v)| \leq C||u||_H \cdot ||v||_H$$

for any $u, v \in H$.

A bilinear form is called Hermitian if $a(u, v) = a(v, u)$ for any $u, v \in H$. A bilinear form is called coercive if there exists a constant $c > 0$ such that

$$a(u, u) \geq c||u||_H^2 \tag{1.2.16}$$

for any $u, v \in H$.

For any bilinear continuous, coercive and Hermitian form $a(u, v)$ on H there exists unique operator $L : H \to H^*$ such that

$$a(u, v) = \langle Lu, v \rangle \tag{1.2.17}$$

for any $v \in H$.

If $u_2 = 0$, the problem (1.2.10)–(1.2.12) is associated with the bilinear form

$$a(u, v) = \int_Q a(\mathbf{x}) \nabla u(\mathbf{x}) \nabla v(\mathbf{x}) d\mathbf{x} \tag{1.2.18}$$

defined on the functional space

$$H = \{v \in H^1(Q) \mid v(\mathbf{x}) = 0 \text{ on } \partial Q \setminus \Gamma\}.$$

The form (1.2.18) satisfies the conditions above if $a(\mathbf{x})$ satisfies the conditions (1.2.13) and (1.2.14).

Then the operator

$$L = \frac{\partial}{\partial x_i} \left(a(\mathbf{x}) \frac{\partial}{\partial x_i} \right)$$

can be treated as an operator acting from H to H^*. The equation (1.2.15) is now valid for any test function in H. Namely, it takes the form

$$\int_Q a(\mathbf{x}) \frac{\partial u}{\partial x_i}(\mathbf{x}) \frac{\partial v}{\partial x_i}(\mathbf{x}) d\mathbf{x} = - \int_Q f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} + \int_\Gamma u_1(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \tag{1.2.19}$$

for any $v \in H$.

The problem (1.2.19) is called the variational form of the problem (1.2.10)–(1.2.12). The equation (1.2.19) is a necessary condition of the minimum of the quadratic functional

$$I(u) = \int_Q a(\mathbf{x}) |\nabla u(\mathbf{x})|^2 d\mathbf{x} + \int_Q f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} - \int_\Gamma u_1(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \tag{1.2.20}$$

on H.

Under conditions (1.2.13) and (1.2.16) there exists a unique $u \in H$ such that (1.2.19) is satisfied (see, e.g., Ekeland and Temam (1976)).

If the solution of the problem (1.2.15) is sufficiently smooth, then it satisfies (1.2.10)–(1.2.12) with $u_2(\mathbf{x}) = 0$, see, e.g., Ladyzhenskaya and Ural'tseva (1968).

1.3 Duality of functional spaces and functionals

We present brief information about the duality of functional spaces and functionals, the Legendre transform and the minimax problem following Ekeland and Temam (1976). Information on convex analysis in finite-dimensional and functional spaces can be found in Rockafellar (1970, 1969) and Ekeland and Temam (1976) (see also the references in the cited books).

1.3.1 Legendre transform

Let H be a Hilbert space. The set of linear functionals defined on H is also a Hilbert space. It is called the dual (conjugate) space with respect to H and denoted H^*.

As usual (Ekeland and Temam, 1976), we denote by $\langle u, u^* \rangle$ the dual coupling of elements $u \in H$ and $u^* \in H^*$ (the value of the linear functional u^* on the element u).

Let H and H^* be two dual Hilbert spaces and F be a functional of H into R. the functional

$$F^*(u^*) = \sup_{u \in H} \{\langle u, u^* \rangle - F(u)\} \tag{1.3.1}$$

that defines a function from H^* is called the conjugate functional of F.

It is known that F^* is a convex functional and if F is a convex functional, $F^{**} = F$ (Rockafellar, 1970).

For $H = \mathbb{R}^N$ (in this case $H^* = \mathbb{R}^N$) the right-hand side of (1.3.1) is known as a Legendre transform. In this case two convex functions $\phi(\mathbf{x})$ and $\phi^*(\mathbf{y})$, related by the equality

$$\phi^*(\mathbf{y}) = \max_{\mathbf{x} \in \mathbb{R}^N} \{\mathbf{xy} - \phi(\mathbf{x})\}, \tag{1.3.2}$$

are called conjugate functions. In (1.3.2) $\mathbf{xy}$ means the scalar product of the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^N$.

If two numbers α and α^* satisfy $\dfrac{1}{\alpha} + \dfrac{1}{\alpha^*} = 1$, the functions $\phi(\mathbf{x}) = \dfrac{1}{\alpha}|\mathbf{x}|^\alpha$ and $\phi^*(\mathbf{y}) = \dfrac{1}{\alpha^*}|\mathbf{y}|^{\alpha^*}$ are conjugate functions (Ekeland and Temam, 1976).

For $\alpha = 2$, (1.3.2) takes the form

$$\frac{1}{2}|\mathbf{x}|^2 = \max_{\mathbf{y} \in \mathbb{R}^N} \left\{ \mathbf{xy} - \frac{1}{2}|\mathbf{y}|^2 \right\}$$

and the functions $\phi(\mathbf{x}) = \dfrac{1}{2}|\mathbf{x}|^2$ and $\phi^*(\mathbf{y}) = \dfrac{1}{2}|\mathbf{y}|^2$ are conjugate functions.

The functions $\phi(\mathbf{x}) = \dfrac{a}{2}|\mathbf{x}|^2$ and $\phi^*(\mathbf{y}) = \dfrac{1}{2a}|\mathbf{y}|^2$ are conjugate functions for $a \neq 0$. The condition for a maximum of the function $\mathbf{xy} - \dfrac{a}{2}|\mathbf{x}|^2$ is $\mathbf{y} - a\mathbf{x} = 0$. Substituting $\mathbf{x} = \dfrac{\mathbf{y}}{a}$ in the analyzed function, we find that its maximum value is $\dfrac{\mathbf{y}^2}{2a}$.

1.3.2 The minimax problem

Let H and Z be Hilbert spaces. We consider a minimization problem

$$I(\phi) \to \min, \phi \in H. \tag{1.3.3}$$