Spectral Analysis for Univariate Time Series

Spectral analysis is widely used to interpret time series collected in diverse areas such as the environmental, engineering and physical sciences. This book covers the statistical theory behind spectral analysis and provides data analysts with the tools needed to transition theory into practical applications. Actual time series from oceanography, metrology, atmospheric science and other areas are used in running examples throughout, to allow clear comparison of how the various methods address questions of interest.

All major nonparametric and parametric spectral analysis techniques are discussed, with particular emphasis on the multitaper method, both in its original formulation involving Slepian tapers and in a popular alternative using sinusoidal tapers. The authors take a unified approach to quantifying the bandwidth of nonparametric spectral estimates, allowing for meaningful comparison among different estimates. An extensive set of exercises allows readers to test their understanding of both the theory and practical analysis of time series. The time series used as examples and R language code for recreating the analyses of the series are available from the book's web site.

DONALD B. PERCIVAL is the author of 75 publications in refereed journals on a variety of topics, including analysis of environmental time series, characterization of instability of atomic clocks and forecasting inundation of coastal communities due to trans-oceanic tsunamis. He is the coauthor (with Andrew Walden) of *Spectral Analysis for Physical Applications: Multitaper and Conventional Univariate Techniques* (Cambridge University Press, 1993) and *Wavelet Methods for Time Series Analysis* (Cambridge University Press, 2000). He has taught graduate-level courses on time series analysis, spectral analysis and wavelets for over 30 years at the University of Washington.

ANDREW T. WALDEN has authored 100 refereed papers in scientific areas including statistics, signal processing, geophysics, astrophysics and neuroscience, with an emphasis on spectral analysis and time series methodology. He worked in geophysical exploration research before joining Imperial College London. He is coauthor (with Donald B. Percival) of *Spectral Analysis for Physical Applications: Multitaper and Conventional Univariate Techniques* (Cambridge University Press,1993) and *Wavelet Methods for Time Series Analysis* (Cambridge University Press, 2000). He has taught many courses including time series, spectral analysis, geophysical data analysis, applied probability and graphical modeling, primarily at Imperial College London, and also at the University of Washington.

CAMBRIDGE SERIES IN STATISTICAL AND PROBABILISTIC MATHEMATICS

This series of high-quality upper-division textbooks and expository monographs covers all aspects of stochastic applicable mathematics. The topics range from pure and applied statistics to probability theory, operations research, optimization, and mathematical programming. The books contain clear presentations of new developments in the field and also of the state of the art in classical methods. While emphasizing rigorous treatment of theoretical methods, the books also contain applications and discussions of new techniques made possible by advances in computational practice.

A complete list of books in the series can be found at www.cambridge.org/statistics. Recent titles include the following:

24. *Random Networks for Communication,* by Massimo Franceschetti and Ronald Meester
25. *Design of Comparative Experiments,* by R. A. Bailey
26. *Symmetry Studies,* by Marlos A. G. Viana
27. *Model Selection and Model Averaging,* by Gerda Claeskens and Nils Lid Hjort
28. *Bayesian Nonparametrics,* edited by Nils Lid Hjort *et al.*
29. *From Finite Sample to Asymptotic Methods in Statistics,* by Pranab K. Sen, Julio M. Singer and Antonio C. Pedrosa de Lima
30. *Brownian Motion,* by Peter Mörters and Yuval Peres
31. *Probability: Theory and Examples (Fourth Edition),* by Rick Durrett
33. *Stochastic Processes,* by Richard F. Bass
34. *Regression for Categorical Data,* by Gerhard Tutz
35. *Exercises in Probability (Second Edition),* by Loïc Chaumont and Marc Yor
36. *Statistical Principles for the Design of Experiments,* by R. Mead, S. G. Gilmour and A. Mead
37. *Quantum Stochastics,* by Mou-Hsiung Chang
38. *Nonparametric Estimation under Shape Constraints*, by Piet Groeneboom and Geurt Jongbloed
39. *Large Sample Covariance Matrices and High-Dimensional Data Analysis,* by Jianfeng Yao, Shurong Zheng and Zhidong Bai
40. *Mathematical Foundations of Infinite-Dimensional Statistical Models,* by Evarist Giné and Richard Nickl
41. *Confidence, Likelihood, Probability,* by Tore Schweder and Nils Lid Hjort
42. *Probability on Trees and Networks*, by Russell Lyons and Yuval Peres
43. *Random Graphs and Complex Networks (Volume 1)*, by Remco van der Hofstad
44. *Fundamentals of Nonparametric Bayesian Inference*, by Subhashis Ghosal and Aad van der Vaart
45. *Long-Range Dependence and Self-Similarity*, by Vladas Pipiras and Murad S. Taqqu
46. *Predictive Statistics*, by Bertrand S. Clarke and Jennifer L. Clarke
47. *High-Dimensional Probability,* by Roman Vershynin
48. *High-Dimensional Statistics,* by Martin J. Wainwright
49. *Probability: Theory and Examples (Fifth Edition)*, by Rick Durrett
50. *Statistical Model-based Clustering and Classification*, by Charles Bouveyron *et al.*
51. *Spectral Analysis for Univariate Time Series*, by Donald B. Percival and Andrew T. Walden

Spectral Analysis for Univariate Time Series

Donald B. Percival
University of Washington

Andrew T. Walden
Imperial College of Science, Technology and Medicine

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107028142
DOI: 10.1017/9781139235723

First published 2020

Printed in the United Kingdom by TJ International, Padstow Cornwall

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Percival, Donald B., author. | Walden, Andrew T., author.
Title: Spectral analysis for univariate time series / Donald B. Percival, Andrew T. Walden.
Description: Cambridge ; New York, NY : Cambridge University Press, 2020. |
Series: Cambridge series on statistical and probabilistic mathematics ;
51 | Includes bibliographical references and indexes.
Identifiers: LCCN 2019034915 (print) | LCCN 2019034916 (ebook) |
ISBN 9781107028142 (hardback) | ISBN 9781139235723 (ebook)
Subjects: LCSH: Time-series analysis. | Spectral theory (Mathematics)
Classification: LCC QA280 .P445 2020 (print) | LCC QA280 (ebook) | DDC 519.5/5–dc23
LC record available at https://lccn.loc.gov/2019034915
LC ebook record available at https://lccn.loc.gov/2019034916

ISBN 978-1-107-02814-2 Hardback

To Niko, Samuel and Evelyn (the next generation of spectral analysts)

Cambridge University Press
978-1-107-02814-2 — Spectral Analysis for Univariate Time Series
Donald B. Percival , Andrew T. Walden
Frontmatter
More Information

Contents

Preface

Spectral analysis is one of the most widely used methods for interpreting time series and has been used in diverse areas including – but not limited to – the engineering, physical and environmental sciences. This book aims to help data analysts in applying spectral analysis to actual time series. Successful application of spectral analysis requires both an understanding of its underlying statistical theory and the ability to transition this theory into practice. To this end, we discuss the statistical theory behind all major nonparametric and parametric spectral analysis techniques, with particular emphasis on the multitaper method, both in its original formulation in Thomson (1982) involving Slepian tapers and in a popular alternative involving the sinusoidal tapers advocated in Riedel and Sidorenko (1995). We then use actual time series from oceanography, metrology, atmospheric science and other areas to provide analysts with examples of how to move from theory to practice.

This book builds upon our 1993 book *Spectral Analysis for Physical Applications: Multitaper and Conventional Univariate Techniques* (also published by Cambridge University Press). The motivations for considerably expanding upon this earlier work include the following.

[1] A quarter century of teaching classes based on the 1993 book has given us new insights into how best to introduce spectral analysis to analysts. In particular we have greatly expanded our treatment of the multitaper method. While this method was a main focus in 1993, we now describe it in a context that more readily allows comparison with one of its main competitors (Welch's overlapped segment averaging).

[2] The core material on nonparametric spectral estimation is in Chapters 6 ("Periodogram and Other Direct Spectral Estimators"), 7 ("Lag Window Spectral Estimators") and 8 ("Combining Direct Spectral Estimators"). These chapters now present these estimators in a manner that allows easier comparison of common underlying concepts such as smoothing, bandwidth and windowing.

[3] There have been significant theoretical advances in spectral analysis since 1993, some of which are of particular importance for data analysts to know about. One that we have already mentioned is a new family of multitapers (the sinusoidal tapers) that was introduced in the mid-1990s and that has much to recommend its use. Another is a new bandwidth measure that allows nonparametric spectral analysis methods to be meaningfully compared. A third is bandwidth selection for smoothing periodograms.

[4] An important topic that was not discussed in the 1993 book is computer-based simulation of time series. We devote Chapter 11 to this topic, with particular emphasis on simulating series whose statistical properties agree with those dictated by nonparametric and parametric spectral analyses of actual time series.

[5] We used software written in Common Lisp to carry out spectral analysis for all the time series used in our 1993 book. Here we have used the popular and freely available R software package to do all the data analysis and to create the content for almost all the figures and tables in the book. We do *not* discuss this software explicitly in the book, but we make it available as a supplement so that data analysts can replicate and build upon our use of spectral analysis. The website for the book gives access to the R software and to information about software in other languages – see "Data, Software and Ancillary Material" on page xx for details.

Finally, a key motivation for us to undertake an expansion has been the gratifying response to the 1993 book from its intended audience.

The following features of this book are worth noting.

[1] We provide a large number of exercises (over 300 in all), some of which are embedded within the chapters, and others, at the ends of the chapters. The embedded exercises challenge readers to verify certain theoretical results in the main text, with solutions in an Appendix that is available on the website for the book (see page xx). The exercises at the end of the chapters are suitable for use in a classroom setting (solutions are available only for instructors). These exercises both expand upon the theory presented and delve into the practical considerations behind spectral analysis.

[2] We use actual time series to illustrate various spectral analysis methods. We do so to encourage data analysts to carefully consider the link between spectral analysis and what questions this technique can address about particular series. In some instances we use the same series with different techniques to allow analysts to compare how well various methods address questions of interest.

[3] We provide a large number of "Comments and Extensions" (C&Es) to the main material. These C&Es appear at the ends of sections when appropriate and provide interesting supplements to the main material; however, readers can skip the C&Es without compromising their ability to follow the main material later on (we have set the C&Es in a slightly smaller font to help differentiate them from the main material). The C&Es cover a variety of ancillary – but valuable – topics such as the Lomb–Scargle periodogram, jackknifing of multitaper spectral estimates, the method of surrogate time series, a periodogram based upon the discrete cosine transform (and its connection to Albert Einstein!) and the degree to which windows designed for one purpose can be used for another.

[4] At the end of most chapters, we provide a comprehensive summary of that chapter. The summaries allow readers to check their understanding of the main points in a chapter and to review the content of a previous chapter when tackling a later chapter. The comprehensive subject index at the end of the book will aid in finding details of interest.

We also note that "univariate" is part of the title of the book because a volume on multivariate spectral analysis is in progress.

Books do not arise in isolation, and ours is no exception. With a book that is twenty-five years in the making, the list of editors, colleagues, students, readers of our 1993 book, friends and relatives who have influenced this book in some manner is so long that thanking a select few individuals here will only be at the price of feeling guilty both now and later on about not thanking many others. Those who are on this list know who you are. We propose to thank you with a free libation of your choice upon our first post-publication meeting (wherever this might happen – near Seattle or London or both or elsewhere!). We do, however, want to explicitly acknowledge financial support through EPSRC Mathematics Platform grant EP/I019111/1. We also thank Stan Murphy (posthumously), Bob Spindel and Jeff Simmen (three generations of directors of the Applied Physics Laboratory, University of Washington) for supplying ongoing discretionary funding without which this and the 1993 book would not exist.

Finally, despite our desire for a book needing no errata list, past experience says this will not happen. Readers are encouraged to contact us about blemishes in the book so that we can make others aware of them (our email addresses are listed with our signatures).

Don Percival
Applied Physics Laboratory
Department of Statistics
University of Washington
dbpercival@gmail.com

Andrew Walden
Department of Mathematics
Imperial College London
atwalden86@gmail.com

Conventions and Notation

- *Important conventions*

(14)	refers to the single displayed equation on page 14
(3a), (3b)	refers to different displayed equations on page 3
Figure 2	refers to the figure on page 2
Table 214	refers to the table on page 214
Exercise [8]	refers to the embedded exercise on page 8 (see the Appendix on the website for an answer)
Exercise [1.3]	refers to the third exercise at the end of Chapter 1
a, $\boldsymbol{a}$ and $\boldsymbol{A}$	refer to a scalar, a vector and a matrix/vector
$S(\cdot)$	refers to a function
$S(f)$	refers to the value of the function $S(\cdot)$ at f
$\{h_t\}$	refers to a sequence of values indexed by t
h_t	refers to a single value of a sequence
α and $\hat{\alpha}$	refer to a parameter and an estimator thereof

In the following lists, the numbers at the end of the brief descriptions are page numbers where more information about – or an example of the use of – an abbreviation or symbol can be found.

- *Abbreviations used frequently*

ACF	autocorrelation function	27
ACLS	approximate conditional least squares	549
ACS	autocorrelation sequence	27
ACVF	autocovariance function	27
ACVS	autocovariance sequence	27
AIC	Akaike's information criterion	494
AICC	AIC corrected for bias	495, 576

AR(p)	pth-order autoregressive process	33, 446
ARMA(p, q)	autoregressive moving average process of order (p, q)	35
BLS	backward least squares	477
C&Es	Comments and Extensions	24
CI	confidence interval	204
CPDF	cumulative probability distribution function	23
dB	decibels, i.e., $10 \log_{10}(\cdot)$	13
DCT–II	discrete cosine transform of type II	184, 217
DFT	discrete Fourier transform	74, 92
DPSS	discrete prolate spheroidal sequence (Slepian sequence)	87, 155
DPSWF	discrete prolate spheroidal wave function	87
ECLS	exact conditional least squares	549
EDOFs	equivalent degrees of freedom	264
EULS	exact unconditional least squares	549
FBLS	forward/backward least squares	477, 555
FFT	fast Fourier transform	92, 94
FIR	finite impulse response	147
FLS	forward least squares	476
FPE	final prediction error	493
GCV	generalized cross-validated	309
GSSM	Gaussian spectral synthesis method	605
Hz	Hertz: 1 Hz = 1 cycle per second	
IID	independent and identically distributed	31
IIR	infinite impulse response	147
LS	least squares	475
LTI	linear time-invariant	132
MA(q)	qth-order moving average process	32
MLE	maximum likelihood estimator or estimate	480
MSE	mean square error	167
MSLE	mean square log error	296
NMSE	normalized mean square error	296
OLS	ordinary least squares	409
PACS	partial autocorrelation sequence	462
PDF	probability density function	24
PSWF	prolate spheroidal wave function	64
RV	random variable	3
SDF	spectral density function	111
SS	sum of squares	467–8, 476–7
SVD	singular value decomposition	565
WOSA	Welch's (or weighted) overlapped segment averaging	414
ZMNL	zero-memory nonlinearity	634

- *Non-Greek notation used frequently*

A_l	real-valued amplitude associated with $\cos(2\pi f_l t\,\Delta_t)$	35, 515
$b(\cdot)$	bias	192, 239, 378
$b^{(B)}(\cdot)$	broad-band bias	378
$b^{(L)}(\cdot)$	local bias	378
$b_k(f)$	weight associated with kth eigenspectrum at frequency f	386
$b_W(\cdot)$	bias due to smoothing window only	256
B_l	real-valued amplitude associated with $\sin(2\pi f_l t\,\Delta_t)$	35, 515
$B_{\mathcal{H}}$	bandwidth of spectral window $\mathcal{H}$	194
B_S	spectral bandwidth	292, 297
B_T	bandwidth measure for $\{X_t\}$ with dominantly unimodal SDF	300
$\tilde{B}_T$	approximately unbiased estimator of B_T	300
$B_{\mathcal{U}}$	bandwidth of spectral window $\mathcal{U}_m$	256
B_W	Jenkins measure of smoothing window bandwidth	251
$\{c_\tau\}$	inverse Fourier transform of $C(\cdot)$ (cepstrum if properly scaled)	301
$C(\cdot)$	log spectral density function	301
$\hat{C}^{(D)}(\cdot)$	log of direct spectral estimator	301
$\hat{C}_m^{(LW)}(\cdot)$	smoothed log of direct spectral estimator	303
C_h	variance inflation factor due to tapering	259, 262
C_l	complex-valued amplitude associated with $\exp(i2\pi f_l t\,\Delta_t)$	108, 519
$d(\cdot,\cdot)$	Kullback–Leibler discrepancy measure (general case)	297
$dZ(\cdot)$	orthogonal increment	109
D_l	real-valued amplitude associated with $\cos(2\pi f_l t\,\Delta_t + \phi_l)$ or with $\exp(i[2\pi f_l t\,\Delta_t + \phi_l])$	35, 511, 517
$\boldsymbol{D}_N$	$N \times N$ diagonal matrix	375
$\mathcal{D}_N(\cdot)$	Dirichlet's kernel	17
$\overrightarrow{e}_t(k)$	observed forward prediction error	467
$\overleftarrow{e}_t(k)$	observed backward prediction error	467
f_k	$k/(N\,\Delta_t)$, member of grid of Fourier frequencies	171, 515
f'_k	$k/(N'\,\Delta_t)$, member of arbitrary grid of frequencies	171
$\tilde{f}_k$	$k/(2N\,\Delta_t)$, member of grid twice as fine as Fourier frequencies	181
f_l	frequency of a sinusoid	35, 511
$f_{\mathcal{N}}$	$1/(2\,\Delta_t)$, Nyquist frequency	82, 122, 512
$F_t(\cdot)$	cumulative probability distribution function	23
$\mathcal{F}(\cdot)$	Fejér's kernel	174, 236
g	Fisher's test statistic for simple periodicity	539
g_F	critical value for Fisher's test statistic g	540
$g(\cdot)$	real- or complex-valued function	53
$g_p(\cdot)$	periodic function	48
$\{g_u\}$	impulse response sequence of a digital filter	143
$g \star g^*(\cdot)$	autocorrelation of deterministic function $g(\cdot)$	72
$g \star h^*(\cdot)$	cross-correlation of deterministic functions $g(\cdot)$ and $h(\cdot)$	72
$\{g * h_t\}$	convolution of sequences $\{g_t\}$ and $\{h_t\}$	99

CAMBRIDGE
Cambridge University Press
978-1-107-02814-2 — Spectral Analysis for Univariate Time Series
Donald B. Percival , Andrew T. Walden
Frontmatter
More Information

$g * h(\cdot)$	convolution of functions $g(\cdot)$ and $h(\cdot)$	67
$G(\cdot)$	Fourier transform of $g(\cdot)$ or transfer function	54, 97, 136, 141
$G_p(\cdot)$	Fourier transform of $\{g_t\}$	74, 99
$\{G_n\}$	Fourier transform of $g_p(\cdot)$	49, 96
$\{G_t\}$	stationary *Gaussian* process	201, 445
$\{h_t\}$	data taper	186
$\{h_{k,t}\}$	kth-order data taper for multitaper estimator	352, 357, 392
$H(\cdot)$	Fourier transform of data taper $\{h_t\}$	186
$\{H_t\}$	Gaussian autoregressive process	445
$\mathcal{H}(\cdot)$	spectral window of direct spectral estimator	186
$\mathcal{H}_k(\cdot)$	spectral window of kth eigenspectrum	352
$\overline{\mathcal{H}}(\cdot)$	spectral window of basic multitaper estimator	353
$\widetilde{\mathcal{H}}(\cdot)$	spectral window of weighted multitaper estimator	353
$\mathcal{HT}\{\cdot\}$	Hilbert transform	114, 562, 579
$J(\cdot)$	scaled Fourier transform of tapered time series	186, 544
$K_{\max}$	maximum number of usable Slepian multitapers	357
$\mathrm{KL}(\cdot)$	Kullback–Leibler discrepancy measure (special case)	297
$L\{x(\cdot)\}$	continuous parameter filter acting on function $x(\cdot)$	133
$L\{x_t\}$	discrete parameter filter acting on sequence $\{x_t\}$	141
$\boldsymbol{L}_N$	$N \times N$ lower triangular matrix with 1's on diagonal	464
m	parameter controlling smoothing in lag window estimator	247
N	sample size	2, 163
N'	integer typically greater than or equal to N	179, 237
N_B	number of blocks in WOSA	414
N_S	block size in WOSA	414
p	order of an autoregressive process or a proportion	33, 189, 204
$\mathbf{P}[A]$	probability that the event A will occur	23
$\mathcal{P}_k$	normalized cumulative periodogram	215
q	order of a moving average process	32, 503
$\boldsymbol{Q}$	weight matrix in quadratic spectral estimator	374
$Q_\nu(p)$	$p \times 100\%$ percentage point of χ^2_ν distribution	265
$\{r_\tau\}$	inverse Fourier transform of $R(\cdot)$	212
R	signal-to-noise ratio	526, 532
$R(\eta)$	correlation of direct spectral estimators at f and $f+\eta$	212
$\{R_t\}$	residual process	550
$\{s_\tau\}$	autocovariance sequence (ACVS)	27, 29
$\{s_\tau^{(\mathrm{BL})}\}$	ACVS for band-limited white noise	379
$\{\hat{s}_\tau^{(\mathrm{D})}\}$	ACVS estimator, inverse Fourier transform of $\{\hat{S}^{(\mathrm{D})}(\cdot)\}$	188
$\{\hat{s}_\tau^{(\mathrm{P})}\}$	"biased" estimator of ACVS	166
$\{\hat{s}_\tau^{(\mathrm{U})}\}$	"unbiased" estimator of ACVS	166
$s(\cdot)$	autocovariance function (ACVF)	27
$S(\cdot)$	spectral density function (SDF)	111
$S_\eta(\cdot)$	SDF of (possibly) colored noise	519

$S^{(\mathrm{I})}(\cdot)$	integrated spectrum or spectral distribution function	110
$S^{(\mathrm{BL})}(\cdot)$	SDF of band-limited white noise	379
$\hat{S}^{(\mathrm{AMT})}(\cdot)$	adaptive multitaper spectral estimator	389
$\hat{S}^{(\mathrm{D})}(\cdot)$	direct spectral estimator	186
$\hat{S}^{(\mathrm{DCT})}(\cdot)$	DCT-based periodogram	217
$\hat{S}^{(\mathrm{DS})}(\cdot)$	discretely smoothed direct spectral estimator	246
$\hat{S}_m^{(\mathrm{DSP})}(\cdot)$	discretely smoothed periodogram	307
$\hat{S}_m^{(\mathrm{LW})}(\cdot)$	lag window spectral estimator	247
$\hat{S}^{(\mathrm{MT})}(\cdot)$	basic multitaper spectral estimator	352
$\hat{S}_k^{(\mathrm{MT})}(\cdot)$	kth eigenspectrum for multitaper estimator	352
$\hat{S}^{(\mathrm{P})}(\cdot)$	periodogram (special case of $\hat{S}^{(\mathrm{D})}(\cdot)$)	170, 188
$\tilde{S}^{(\mathrm{P})}(\cdot)$	periodogram of shifted time series or rescaled periodogram	184, 222, 240
$\hat{S}^{(\mathrm{PC})}(\cdot)$	postcolored spectral estimator	198
$\hat{S}^{(\mathrm{Q})}(\cdot)$	quadratic spectral estimator	374
$\hat{S}^{(\mathrm{WMT})}(\cdot)$	weighted multitaper spectral estimator	352
$\hat{S}^{(\mathrm{WOSA})}(\cdot)$	WOSA spectral estimator	414
$\hat{S}^{(\mathrm{YW})}(\cdot)$	Yule–Walker spectral estimator	451
t	actual time (continuous) or a unitless index (discrete)	22, 74
t_λ	critical value for Siegel's test statistic	541
T_λ	Siegel's test statistic	541
$U_k(\cdot; N, W)$	discrete prolate spheroidal wave function of order k	87
$\mathcal{U}_m(\cdot)$	spectral window of $\hat{S}^{(\mathrm{LW})}(\cdot)$	255
$\boldsymbol{v}_k(N, W)$	vector with portion of DPSS, order k	87
$\{v_{m,\tau}\}$	nontruncated version of lag window $\{w_{m,\tau}\}$	247
$V_m(\cdot)$	design window (Fourier transform of $\{v_{m,\tau}\}$)	247
$\{w_{m,\tau}\}$	lag window (truncated version of $\{v_{m,\tau}\}$)	247
$\mathrm{width_a}\{\cdot\}$	autocorrelation width	73
$\mathrm{width_e}\{\cdot\}$	equivalent width	58
$\mathrm{width_{hp}}\{\cdot\}$	half-power width	192
$\mathrm{width_v}\{\cdot\}$	variance width	60, 192
W	DPSS half-bandwidth, regularization half-bandwidth	65, 377
$W_m(\cdot)$	smoothing window	247–8
$x_0, \ldots, x_{N-1}$	time series realization or deterministic series	2
$X_0, \ldots, X_{N-1}$	sequence of random variables	3
$\{X_t\}$	real-valued discrete parameter stochastic process	22
$\{X(t)\}$	real-valued continuous parameter stochastic process	22
$\{X_{j,t}\}$	jth real-valued discrete parameter stochastic process	23
$\{X_j(t)\}$	jth real-valued continuous parameter stochastic process	23
$\overline{X}$	sample mean (arithmetic average) of $X_0, \ldots, X_{N-1}$	164
$\overrightarrow{X}_t(k)$	best (forward) linear predictor of X_t given $X_{t-1}, \ldots, X_{t-k}$	452
$\overleftarrow{X}_t(k)$	best (backward) linear predictor of X_t given $X_{t+1}, \ldots, X_{t+k}$	455
$\{Y_t\}$	real-valued discrete parameter stochastic process (refers to an AR process in Chapter 9)	23, 445

$\{Z_t\}$	complex-valued discrete parameter stochastic process	23, 29
$\{Z(t)\}$	complex-valued continuous parameter stochastic process	23
$\{Z(f)\}$	orthogonal process	109

• *Greek notation used frequently*

α	intercept term in linear model, scalar, level of significance or exponent of power law	39, 43, 215, 327
$\alpha^2(N)$	fraction of sequence's energy lying in $0, \ldots, N-1$	85
$\alpha^2(T)$	fraction of function's energy lying in $[-T/2, T/2]$	62
β	slope term in linear model	39
$\beta^{(\mathrm{B})}\{\cdot\}$	indicator of broad-band bias in $\hat{S}^{(\mathrm{Q})}(\cdot)$	379
$\beta^{(\mathrm{L})}\{\cdot\}$	indicator of magnitude of local bias in $\hat{S}^{(\mathrm{Q})}(\cdot)$	378
β_W	Grenander's measure of smoothing window bandwidth	251
$\beta^2(W)$	fraction of function's energy lying in $[-W, W]$	62, 85
$\beta^2_{\mathcal{H}}$	indicator of bias in $\hat{S}^{(\mathrm{D})}(\cdot)$	391
γ	quadratic term in linear model or Euler's constant	46, 210
$\Gamma(\cdot)$	gamma function	440
$\boldsymbol{\Gamma}$	covariance matrix (typically for AR(p) process)	450, 464
Δ_{f}	spacing in frequency	206
Δ_{t}	spacing in time (sampling interval)	74, 81–2, 122
$\{\epsilon_t\}$	white noise or innovation process	32, 446
$\overrightarrow{\epsilon}_t(k)$	forward prediction error: $X_t - \overrightarrow{X}_t(k)$	453
$\overleftarrow{\epsilon}_t(k)$	backward prediction error: $X_t - \overleftarrow{X}_t(k)$	455
η	equivalent degrees of freedom of a time series	298
$\{\eta_t\}$	zero mean stationary noise process (not necessarily white)	518
$\theta(\cdot)$	phase function corresponding to transfer function $G(\cdot)$	136
θ	coefficient of an MA(1) process	43
$\theta_{q,1}, \ldots, \theta_{q,q}$	coefficients of an MA(q) process	32
$\vartheta_{q,0}, \ldots, \vartheta_{q,q}$	coefficients of an MA(q) process ($\vartheta_{q,0} = 1$ and $\vartheta_{q,j} = -\theta_{q,j}$)	594
λ	constant to define different logarithmic scales	301
$\lambda_k(c)$	eigenvalue associated with PSWF, order k	64
$\lambda_k(N, W)$	eigenvalue associated with DPSWF, order k	86
μ	expected value of a stationary process	27
ν	degrees of freedom associated with RV χ^2_ν	202, 264
$\{\rho_\tau\}$	autocorrelation sequence (ACS)	27
$\rho(\cdot)$	autocorrelation function (ACF)	27
σ^2	variance	27
σ^2_ϵ	white noise variance or innovation variance	32, 404
σ^2_η	variance of noise process $\{\eta_t\}$	519
σ^2_k	mean square linear prediction error for $\overrightarrow{X}_t(k)$ or $\overleftarrow{X}_t(k)$	453, 455
σ^2_p	innovation variance for an AR(p) process	446
$\hat{\sigma}^2_p$	estimator of σ^2_p	485
$\bar{\sigma}^2_p$	Burg estimator of σ^2_p	468

$\tilde{\sigma}_p^2$	Yule–Walker estimator of σ_p^2	451, 458
$\boldsymbol{\Sigma}$	covariance matrix	28
τ	lag value	27
ϕ_l	phase of a sinusoid	35, 511
ϕ	coefficient of an AR(1) process	44
$\phi_{p,1}, \ldots, \phi_{p,p}$	coefficients of an AR(p) process	33, 446
$\hat{\phi}_{p,1}, \ldots, \hat{\phi}_{p,p}$	estimators of AR(p) coefficients	485
$\bar{\phi}_{p,1}, \ldots, \bar{\phi}_{p,p}$	Burg estimators of AR(p) coefficients	466, 505
$\tilde{\phi}_{p,1}, \ldots, \tilde{\phi}_{p,p}$	Yule–Walker estimators of AR(p) coefficients	451, 458, 505
$\varphi_{2p,1}, \ldots, \varphi_{2p,2p}$	coefficients of a pseudo-AR($2p$) process	553
$\boldsymbol{\Phi}_p, \hat{\boldsymbol{\Phi}}_p, \tilde{\boldsymbol{\Phi}}_p$	$[\phi_{p,1}, \ldots, \phi_{p,p}]^T$, $[\hat{\phi}_{p,1}, \ldots, \hat{\phi}_{p,p}]^T$, $[\tilde{\phi}_{p,1}, \ldots, \tilde{\phi}_{p,p}]^T$	450, 485, 451
$\Phi^{-1}(p)$	$p \times 100\%$ percentage point of standard Gaussian distribution	265
χ_ν^2	chi-square RV with ν degrees of freedom	37, 202
$\psi(\cdot)$	digamma function	210, 296
$\psi'(\cdot)$	trigamma function	296
$\psi_k(\cdot; c)$	prolate spheroidal wave function (PSWF), order k	64
ω	angular frequency	8, 119

- *Standard mathematical symbols*

e	base for natural logarithm ($2.718282\cdots$)	17
i	$\sqrt{-1}$	17
$\log(\cdot)$, $\log_{10}(\cdot)$	log base e, log base 10	
$\approx$	approximately equal to	
$\doteq$	equal at given precision, e.g., $\pi \doteq 3.1416$	
$\stackrel{\text{def}}{=}$	equal by definition	23–4
$\stackrel{\text{d}}{=}$	equal in distribution	203
$\stackrel{\text{ms}}{=}$	equal in mean square sense	49
$E\{\cdot\}$	expectation operator	24
$\text{var}\{\cdot\}$	variance operator	24–5, 27
$\text{cov}\{\cdot,\cdot\}$	covariance operator	24–5, 27
$\text{corr}\{\cdot,\cdot\}$	correlation operator	27
z^*	complex conjugate of z	25
$\Re(z)$	real part of complex-valued number z	61, 551
$\Im(z)$	imaginary part of complex-valued number z	551
$\arg(z)$	argument of complex-valued number z	54
$*$	convolution operator	67, 96, 98–9, 101
$\star$	cross-correlation operator	72, 96, 98–9, 101
$\longleftrightarrow$	Fourier transform pair relationship	69, 96–7, 99–100
$\boldsymbol{a}^T$ and $\boldsymbol{Q}^T$	transpose of vector $\boldsymbol{a}$ and of matrix $\boldsymbol{Q}$	28, 374
$\boldsymbol{Z}^H$ and $\boldsymbol{Q}^H$	Hermitian transpose of vector $\boldsymbol{Z}$ and of matrix $\boldsymbol{Q}$	374
$\boldsymbol{I}_N$	$N \times N$ identity matrix	366
$\text{tr}\{\boldsymbol{Q}\}$	trace of matrix $\boldsymbol{Q}$	376

$\lvert \boldsymbol{\Gamma}_N \rvert$	determinant of matrix $\boldsymbol{\Gamma}_N$	480
$\boldsymbol{R}^{\#}$	generalized inverse of matrix $\boldsymbol{R}$	566
$\langle \cdot, \cdot \rangle$	inner product	470
$\lvert\lvert \cdot \rvert\rvert^2$	squared norm	476
$\mathbb{R}$	set of all real-valued numbers, i.e., $\{t : -\infty < t < \infty\}$	22
$\mathbb{Z}$	set of all integers, i.e., $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$	17
$L^2(\cdot)$	set of square integrable functions over specified domain	54–5
$\in, \notin$	contained in, not contained in	
δ_n	Kronecker delta function	44, 65
$\delta(\cdot)$	Dirac delta function	74, 120
$\mathrm{sinc}\,(\cdot)$	sinc function, i.e., $\sin(\pi t)/(\pi t)$	63
$\lfloor x \rfloor$	greatest integer $\le x$, e.g., $\lfloor \pi \rfloor = 3$ and $\lfloor 3 \rfloor = 3$	7–8
$(a)_+$	positive part, i.e., $\max\{a, 0\}$	541
mod	modulo operator, e.g., $5 \bmod 4 = 1$	45, 101
$O(\cdot)$	$f(x) = O(g(x))$ as $x \to 0$ if $\lvert f(x)/g(x) \rvert \le C$ for constant C	175

Data, Software and Ancillary Material

The website for this book is currently at

http://faculty.washington.edu/dbp/sauts.html

(alternatively go to www.cambridge.org/9781107028142 – this is maintained by Cambridge University Press and should have both a description of the book and a link to the current location for the book's website). The website gives access to

- an Appendix with answers to all the exercises embedded within Chapters 1 to 11;
- almost all the time series used as examples in the book;
- software in the R language for recreating the content in the bulk of the figures and tables in each chapter (and updates on the status of software in other languages);
- the current errata sheet (we encourage readers who spot errata to contact us via email at dbpercival@gmail.com and/or atwalden86@gmail.com);
- information about how to obtain a solutions guide for the exercises at the end of each chapter (this guide is *only* for instructors who are using the book as course material); and
- PDF files for the bulk of the figures and tables in the book (these are intended to help prepare course and seminar material, but please note that all figures and tables are the copyright of Cambridge University Press and must not be further distributed or used without written permission).