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Excerpt
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16

The spaces $\mathcal{M}(A)$ and $\mathcal{H}(A)$

In this chapter, we introduce the notion of *complementary space*, which generalizes the classic geometric notion of orthogonal complement. This notion of complementary space is central in the theory of $\mathcal{H}(b)$ spaces. In Section 16.1, we study the bounded (contractively or isometrically) embeddings. This leads to the definition of $\mathcal{M}(A)$ spaces. Then, in Section 16.2, we characterize the relations between two $\mathcal{M}(A)$ spaces. In Section 16.3, we describe the linear functional on $\mathcal{M}(A)$. In Section 16.4, we give our first definition of complementary space based on an operatorial point of view. As we will see in the next chapter, this operatorial point of view seems particularly interesting in the context of $\mathcal{H}(b)$ spaces and Toeplitz operators. In Section 16.5, we describe the relation between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$. This relation, though very simple, is probably one of the most useful results in the theory of $\mathcal{H}(b)$ spaces. The overlapping space is introduced and described in Section 16.6. In Sections 16.7 and 16.8, we give useful results concerning some decomposition of $\mathcal{M}(A)$ and $\mathcal{H}(A)$ spaces. In Section 16.9, we introduce our second definition of complementary space and show that it coincides with the first one. Finally, in the last section, we show how the Julia operator can be used to connect this notion of complementary spaces to the more familiar geometric structure of orthogonal complements.

16.1 The space $\mathcal{M}(A)$

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $\mathcal{H}_1 \subset \mathcal{H}_2$. We do not necessarily assume that $\mathcal{H}_1$ inherits the Hilbert structure of $\mathcal{H}_2$. They can have different Hilbert space structures. The assumption $\mathcal{H}_1 \subset \mathcal{H}_2$ ensures that the inclusion mapping

$$\begin{array}{rccc} i: & \mathcal{H}_1 & \longrightarrow & \mathcal{H}_2 \\ & x & \longmapsto & x \end{array}$$

is well defined. If this mapping is bounded, i.e. if there is a constant $c > 0$ such that

$$\|x\|_{\mathcal{H}_2} \le c\,\|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1), \tag{16.1}$$

we say that $\mathcal{H}_1$ is *boundedly* contained in $\mathcal{H}_2$ and write $\mathcal{H}_1 \Subset \mathcal{H}_2$. If the mapping i is a contraction, i.e. $c \le 1$, we say that $\mathcal{H}_1$ is *contractively* included in $\mathcal{H}_2$ and write $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$. Finally, if

$$\|x\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1),$$

we say that $\mathcal{H}_1$ is *isometrically* contained in $\mathcal{H}_2$. If it happens that the set identity $\mathcal{H}_1 = \mathcal{H}_2$ holds and, moreover, $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same Hilbert space structure, i.e. $\|x\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1}$ for all possible x, then we write $\mathcal{H}_1 \doteq \mathcal{H}_2$. It is important to distinguish between the set identity $\mathcal{H}_1 = \mathcal{H}_2$ and the Hilbert space identity $\mathcal{H}_1 \doteq \mathcal{H}_2$.

A very special case of the above phenomenon is when $\mathcal{H}_1$ is a closed subspace of $\mathcal{H}_2$ and inherits its Hilbert space structure. In this case, $\mathcal{H}_1$ is isometrically embedded inside $\mathcal{H}_2$. In the next section, we will look at this phenomenon from a slightly different angle.

The inequality (16.1) reveals some facts about the topologies of $\mathcal{H}_1$ and $\mathcal{H}_2$. If $\mathcal{E}$ is a closed (or open) subset of $\mathcal{H}_2$, then $\mathcal{E} \cap \mathcal{H}_1$ is closed (or open) in $\mathcal{H}_1$ with respect to the topology of $\mathcal{H}_1$. However, the topology of $\mathcal{H}_1$ is usually richer. In other words, the topology of $\mathcal{H}_1$ is finer than the topology it inherits from $\mathcal{H}_2$. That is why, if Λ is a continuous function on $\mathcal{H}_2$, then its restriction to $\mathcal{H}_1$ remains continuous. We will treat this fact in more detail in Section 16.3. As a special case, if $\mathcal{E} \subset \mathcal{H}_1 \subset \mathcal{H}_2$ is closed in $\mathcal{H}_2$, then $\mathcal{E}$ is also closed in $\mathcal{H}_1$. However, if $\mathcal{E}$ is closed in $\mathcal{H}_1$, we cannot conclude that it is also closed in $\mathcal{H}_2$. The following result reveals the relation between different closures of a set in $\mathcal{H}_1$.

Lemma 16.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, assume that $\mathcal{H}_1$ is boundedly embedded into $\mathcal{H}_2$, and let $\mathcal{E} \subset \mathcal{H}_1$. Then*

$$\mathrm{Clos}_{\mathcal{H}_2}(\mathrm{Clos}_{\mathcal{H}_1}\mathcal{E}) = \mathrm{Clos}_{\mathcal{H}_2}\mathcal{E}.$$

Proof For simplicity, put $\mathcal{F} = \mathrm{Clos}_{\mathcal{H}_1}\mathcal{E}$. Since $\mathcal{E} \subset \mathcal{F}$, we have

$$\mathrm{Clos}_{\mathcal{H}_2}\mathcal{E} \subset \mathrm{Clos}_{\mathcal{H}_2}\mathcal{F}.$$

To prove the converse, let $x \in \mathrm{Clos}_{\mathcal{H}_2}\mathcal{F}$ and fix any $\varepsilon > 0$. Then there exists $y \in \mathcal{F}$ such that $\|x - y\|_{\mathcal{H}_2} \le \varepsilon/2$. But, since $y \in \mathcal{F}$ and $\mathcal{F} = \mathrm{Clos}_{\mathcal{H}_1}\mathcal{E}$, there exists $z \in \mathcal{E}$ such that $\|y - z\|_{\mathcal{H}_1} \le \varepsilon/2C$, where C is the constant of embedding of $\mathcal{H}_1$ into $\mathcal{H}_2$, i.e.

$$\|x\|_{\mathcal{H}_2} \le C\,\|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Therefore, we have $\|y-z\|_{\mathcal{H}_2} \le \varepsilon/2$ and then

$$\|x-z\|_{\mathcal{H}_2} \le \|x-y\|_{\mathcal{H}_2} + \|y-z\|_{\mathcal{H}_2} \le \varepsilon.$$

Therefore, $x \in \mathrm{Clos}_{\mathcal{H}_2}\mathcal{E}$. □

Suppose that $\mathcal{H}_1$ is a Hilbert space, $\mathcal{H}_2$ is a set and $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a set bijection between $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the map A can be served to transfer the Hilbert space structure of $\mathcal{H}_1$ to $\mathcal{H}_2$. It is enough to define

$$\langle Ax, Ay\rangle_{\mathcal{H}_2} = \langle x, y\rangle_{\mathcal{H}_1} \tag{16.2}$$

for all $x, y \in \mathcal{H}_1$. The algebraic operations on $\mathcal{H}_2$ are defined similarly. If $\mathcal{H}_2$ is a linear space and A is an algebraic isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$, the latter requirement is already fulfilled. In this case, (16.2) puts an inner product, maybe a new one, on $\mathcal{H}_2$.

The above construction sounds very elementary. Nevertheless, it has profound consequences. In fact, it is the main ingredient in the definition of $\mathcal{H}(b)$ spaces. To move in this direction, suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and that $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. By the first homomorphism theorem, the operator A induces an isomorphism between the quotient space $\mathcal{H}_1/\ker A$ and $\mathcal{R}(A)$. Hence, by (16.2), the identity

$$\langle Ax, Ay\rangle_{\mathcal{R}(A)} = \langle x + \ker A,\ y + \ker A\rangle_{\mathcal{H}_1/\ker A} \qquad (x, y \in \mathcal{H}_1) \tag{16.3}$$

gives a Hilbert space structure on $\mathcal{R}(A)$. We denote this Hilbert space by $\mathcal{M}(A)$. The norm of $x + \ker A$ in $\mathcal{H}_1/\ker A$ is originally defined by

$$\|x + \ker A\|_{\mathcal{H}_1/\ker A} = \inf_{z \in \ker A} \|x + z\|_{\mathcal{H}_1}.$$

But, for each $z \in \ker A$,

$$\begin{aligned}\|x+z\|^2_{\mathcal{H}_1} &= \|P_{(\ker A)^\perp}x + (z + P_{\ker A}x)\|^2_{\mathcal{H}_1}\\ &= \|P_{(\ker A)^\perp}x\|^2_{\mathcal{H}_1} + \|z + P_{\ker A}x\|^2_{\mathcal{H}_1},\end{aligned}$$

and thus we easily see that

$$\|x + \ker A\|_{\mathcal{H}_1/\ker A} = \|P_{(\ker A)^\perp}x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Hence, by the polarization identity (1.16), we have

$$\langle x+\ker A,\ y+\ker A\rangle_{\mathcal{H}_1/\ker A} = \langle P_{(\ker A)^\perp}x,\ P_{(\ker A)^\perp}y\rangle_{\mathcal{H}_1} \qquad (x, y \in \mathcal{H}_1).$$

Moreover, by (1.27),

$$\langle P_{(\ker A)^\perp}x,\ P_{(\ker A)^\perp}y\rangle_{\mathcal{H}_1} = \langle x,\ P_{(\ker A)^\perp}y\rangle_{\mathcal{H}_1} = \langle P_{(\ker A)^\perp}x,\ y\rangle_{\mathcal{H}_1}.$$

Therefore, the definition (16.3) reduces to

$$\begin{aligned} \langle Ax, Ay\rangle_{\mathcal{M}(A)} &= \langle P_{(\ker A)^\perp}x,\ P_{(\ker A)^\perp}y\rangle_{\mathcal{H}_1} \\ &= \langle x,\ P_{(\ker A)^\perp}y\rangle_{\mathcal{H}_1} \\ &= \langle P_{(\ker A)^\perp}x,\ y\rangle_{\mathcal{H}_1} \end{aligned} \tag{16.4}$$

for each $x, y \in \mathcal{H}_1$. In particular, for each $x \in \mathcal{H}_1$,

$$\|Ax\|_{\mathcal{M}(A)} = \|P_{(\ker A)^\perp}x\|_{\mathcal{H}_1}. \tag{16.5}$$

Moreover, if at least one of x or y is orthogonal to $\ker A$, then, by (16.4),

$$\langle Ax, Ay\rangle_{\mathcal{M}(A)} = \langle x, y\rangle_{\mathcal{H}_1}. \tag{16.6}$$

The rather trivial inequality

$$\|Ax\|_{\mathcal{M}(A)} \le \|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1), \tag{16.7}$$

which is a direct consequence of (16.5), will also be frequently used. The preceding formulas should be kept in mind throughout the text.

On $\mathcal{R}(A)$ we now have two inner products. One is inherited from $\mathcal{H}_2$ and the new one imposed by A. In the following, when we write $\mathcal{M}(A)$ we mean that $\mathcal{R}(A)$ is endowed with the latter structure. If this is not the case, we will explicitly mention which structure is considered on $\mathcal{R}(A)$. Let us explore the relation between these two structures. Since A is a bounded operator, we have

$$\|Ax\|_{\mathcal{H}_2} = \|AP_{(\ker A)^\perp}x\|_{\mathcal{H}_2} \le \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}\,\|P_{(\ker A)^\perp}x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

Therefore, by (16.5),

$$\|Ax\|_{\mathcal{H}_2} \le \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}\,\|Ax\|_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1). \tag{16.8}$$

This inequality means that the inclusion map

$$\begin{aligned} i:\ \mathcal{M}(A) &\longrightarrow \mathcal{H}_2 \\ w &\longmapsto w \end{aligned}$$

is continuous and its norm is at most $\|A\|$. In fact, by (16.7),

$$\|Ax\|_{\mathcal{H}_2} \le \|i\|\,\|Ax\|_{\mathcal{M}(A)} \le \|i\|\,\|x\|_{\mathcal{H}_1} \qquad (x \in A).$$

Thus, considering (16.8), we deduce that

$$\|i\|_{\mathcal{L}(\mathcal{M}(A),\mathcal{H}_2)} = \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}. \tag{16.9}$$

Moreover,

$$i^* = AA^*. \tag{16.10}$$

Indeed, let $y \in \mathcal{H}_2$ and $Ax \in \mathcal{M}(A)$, with $x \in \mathcal{H}_1$ and $x \perp \ker A$. Then we have

$$\langle Ax, i^*y\rangle_{\mathcal{M}(A)} = \langle Ax, y\rangle_{\mathcal{H}_2} = \langle x, A^*y\rangle_{\mathcal{H}_1} = \langle Ax, AA^*y\rangle_{\mathcal{M}(A)},$$

which proves that $i^*y = AA^*y$. We will see in Section 16.8 that, in a sense, the operator i^* plays the role of an orthogonal projection of $\mathcal{H}_2$ onto $\mathcal{M}(A)$.

If A is invertible, then the relations (16.7), (16.8) and

$$\|x\|_{\mathcal{H}_1} = \|A^{-1}Ax\|_{\mathcal{H}_1} \le \|A^{-1}\|\,\|Ax\|_{\mathcal{H}_2}$$

imply that the norms in $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{M}(A)$ (which as a set is equal to $\mathcal{H}_2$) are equivalent, i.e.

$$\|x\|_{\mathcal{H}_1} \asymp \|Ax\|_{\mathcal{H}_2} \asymp \|Ax\|_{\mathcal{M}(A)}. \tag{16.11}$$

If A is a bounded operator, the above construction puts $\mathcal{M}(A)$ boundedly inside $\mathcal{H}_2$. If A is a contraction, i.e. $\|A\| \le 1$, then $\mathcal{M}(A)$ is contractively contained in $\mathcal{H}_2$; and if $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{H}_2}$, $w \in \mathcal{M}(A)$, then $\mathcal{M}(A)$ is isometrically contained in $\mathcal{H}_2$. Based on the conventions made in Section 16.1, we emphasize that, for $A, B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, the notation $\mathcal{M}(A) \doteq \mathcal{M}(B)$ means not only that the algebraic equality $\mathcal{M}(A) = \mathcal{M}(B)$ holds, but also that the Hilbert space structures coincide, i.e.

$$\langle w_1, w_2\rangle_{\mathcal{M}(A)} = \langle w_1, w_2\rangle_{\mathcal{M}(B)}$$

for all possible elements w_1 and w_2. Clearly, in the light of the polarization identity, the latter is equivalent to

$$\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(B)}$$

for all possible elements w.

The relation (16.5) contains all the information regarding the definition of the structure of $\mathcal{M}(A)$. In short, the structure of $\mathcal{M}(A)$ is the same as that of $\mathcal{H}_1/\ker A$. This fact is explained in another language in the following result.

Theorem 16.2 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and define*

$$\begin{array}{rccc} \mathbb{A}: & \mathcal{H}_1 & \longrightarrow & \mathcal{M}(A) \\ & x & \longmapsto & Ax. \end{array}$$

Then $\mathbb{A}$ is a bounded operator, i.e. $\mathbb{A} \in \mathcal{L}(\mathcal{H}_1, \mathcal{M}(A))$, and, moreover, $\mathbb{A}^*$ is an isometry on $\mathcal{M}(A)$.

Proof The inequality (16.7) can be rewritten as

$$\|\mathbb{A}x\|_{\mathcal{M}(A)} = \|Ax\|_{\mathcal{M}(A)} \le \|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1).$$

This means that $\mathbb{A}$ is a bounded operator. In order to show that $\mathbb{A}^*$ is an isometry on $\mathcal{M}(A)$, by Corollary 7.23, it is enough to show that $\mathbb{A}$ is a surjective partial isometry. That $\mathbb{A}$ is surjective is a trivial consequence of the definition of $\mathcal{M}(A)$. Moreover, $\ker \mathbb{A} = \ker A$. Hence, by (16.5),

$$\|\mathbb{A}x\|_{\mathcal{M}(A)} = \|Ax\|_{\mathcal{M}(A)} = \|P_{(\ker A)^\perp}x\|_{\mathcal{H}_1} = \|P_{(\ker \mathbb{A})^\perp}x\|_{\mathcal{H}_1}$$

for each $x \in \mathcal{H}_1$. Thus, $\mathbb{A}$ is a partial isometry (see the original definition (7.14)). □

The definition of spaces $\mathcal{M}(A)$ is closely related to the notion of bounded embeddings introduced at the beginning of this section. Indeed, if $\mathcal{M}$ is a Hilbert space that is boundedly contained in another Hilbert space $\mathcal{H}$, then the inclusion map

$$\begin{array}{rccc} i: & \mathcal{M} & \longrightarrow & \mathcal{H} \\ & x & \longmapsto & x \end{array}$$

is bounded from $\mathcal{M}$ into $\mathcal{H}$. Now, since for any $x \in \mathcal{M} = \mathcal{M}(i)$, we have

$$\|x\|_{\mathcal{M}(i)} = \|i(x)\|_{\mathcal{M}(i)} = \|x\|_{\mathcal{M}},$$

the space $\mathcal{M}$ coincides with $\mathcal{M}(i)$, that is

$$\mathcal{M} \doteq \mathcal{M}(i).$$

Conversely, if $\mathcal{M} \doteq \mathcal{M}(A)$, where $A : \mathcal{H}_1 \longrightarrow \mathcal{H}$ is bounded, then $\mathcal{M}$ is boundedly contained in $\mathcal{H}$. Thus, we get the following result.

Theorem 16.3 *Let $\mathcal{M}$ and $\mathcal{H}$ be two Hilbert spaces. Then the following assertions are equivalent.*

(i) *The space $\mathcal{M}$ is boundedly contained in $\mathcal{H}$ (respectively contractively; respectively isometrically).*
(ii) *There exists a bounded operator $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ (respectively a contraction; respectively an isometry) such that*

$$\mathcal{M} \doteq \mathcal{M}(A). \tag{16.12}$$

In the next section, we examine the problem of uniqueness in the representation of $\mathcal{M}$ given by (16.12). See also Exercise 16.2.2.

The following result shows that, if $A \in \mathcal{L}(\mathcal{H})$ is an orthogonal projection, then in fact we do not obtain a new structure on $\mathcal{M}(A)$. The Hilbert space structure of $\mathcal{M}(A)$ is precisely the one it has in the first place as a closed subspace of $\mathcal{H}$.

Lemma 16.4 *Let M be a closed subspace of $\mathcal{H}$, and let $P_M \in \mathcal{L}(\mathcal{H})$ denote the orthogonal projection on M. Then*

$$\mathcal{M}(P_M) \doteq M,$$

i.e. $\mathcal{M}(P_M) = M$ and $\|w\|_{\mathcal{M}(P_M)} = \|w\|_{\mathcal{H}}$ for all $w \in M$.

Proof The identity $\mathcal{M}(P_M) = M$ is an immediate consequence of the definition of an orthogonal projection. Remember that $\ker P_M = M^\perp$, and since M is closed, $(M^\perp)^\perp = M$. Hence, by (16.5),

$$\|P_M x\|_{\mathcal{M}(P_M)} = \|P_{(\ker P_M)^\perp} x\|_{\mathcal{H}} = \|P_M x\|_{\mathcal{H}} \qquad (x \in \mathcal{H}_1).$$ □

Lemma 16.5 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$. Then*

$$\|Bw\|_{\mathcal{M}(BA)} \le \|w\|_{\mathcal{M}(A)} \qquad (w \in \mathcal{M}(A)).$$

Proof It is clear that $B\,\mathcal{M}(A) \subset \mathcal{M}(BA)$. Put $w = Ax$, $x \in \mathcal{H}_1$. Hence, by (16.5),

$$\|Bw\|_{\mathcal{M}(BA)} = \|P_{(\ker BA)^\perp} x\|_{\mathcal{H}_1} \quad \text{and} \quad \|w\|_{\mathcal{M}(A)} = \|P_{(\ker A)^\perp} x\|_{\mathcal{H}_1}.$$

But, since $\ker BA \supset \ker A$, we have

$$\|P_{(\ker BA)^\perp} x\|_{\mathcal{H}_1} \le \|P_{(\ker A)^\perp} x\|_{\mathcal{H}_1}.$$

Therefore, we deduce that $\|Bw\|_{\mathcal{M}(BA)} \le \|w\|_{\mathcal{M}(A)}$. □

Exercises

Exercise 16.1.1 Let $\mathcal{H}$ be a set endowed with two inner products whose corresponding norms are complete and equivalent, i.e.

$$c\,\|x\|_1 \le \|x\|_2 \le C\,\|x\|_1 \qquad (x \in \mathcal{H}),$$

where c and C are positive constants. Show that $(\mathcal{H}, \langle\cdot,\cdot\rangle_1)$ is boundedly contained in $(\mathcal{H}, \langle\cdot,\cdot\rangle_2)$, and vice versa.

Exercise 16.1.2 Let $(X, \mathcal{A})$ be a measurable space, and let μ and ν be two positive measures on the σ-algebra $\mathcal{A}$. Suppose that

$$\mu(E) \le \nu(E) \tag{16.13}$$

for all $E \in \mathcal{A}$. Show that $L^2(\nu)$ is contractively contained in $L^2(\mu)$.

Hint: The assumption (16.13) can be rewritten as

$$\int_X \chi_E \, d\mu \le \int_X \chi_E \, d\nu,$$

where χ_E is the characteristic function of E. Take linear combinations with positive coefficients, and then apply the monotone convergence theorem to obtain

$$\int_X \varphi \, d\mu \le \int_X \varphi \, d\nu$$

for all positive measurable functions φ. Hence, deduce $\|f\|_{L^2(\mu)} \le \|f\|_{L^2(\nu)}$.

Exercise 16.1.3 Let $\varphi \in L^\infty(\mathbb{T})$, and consider the multiplication operator

$$\begin{array}{rccc} M_\varphi: & L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ & f & \longmapsto & \varphi f, \end{array}$$

which was studied in Section 7.2. Show that

$$\|\varphi f\|_{\mathcal{M}(M_\varphi)} = \left(\frac{1}{2\pi} \int_{\mathbb{T}\setminus E} |f(e^{it})|^2 \, dt \right)^{1/2} \qquad (f \in L^2(\mathbb{T}))$$

and that

$$\langle \varphi f, \varphi g \rangle_{\mathcal{M}(M_\varphi)} = \frac{1}{2\pi} \int_{\mathbb{T}\setminus E} f(e^{it}) \, \overline{g(e^{it})} \, dt \qquad (f, g \in L^2(\mathbb{T})),$$

where $E = \{\zeta \in \mathbb{T} : \varphi(\zeta) = 0\}$. The first identity reveals that $\mathcal{M}(M_\varphi) = \varphi L^2(\mathbb{T})$ is contractively contained in $L^2(\mathbb{T})$. Under what condition is $\mathcal{M}(M_\varphi)$ isometrically contained in $L^2(\mathbb{T})$?

Exercise 16.1.4 Let Θ be an inner function for the open unit disk, and let

$$\begin{array}{rccc} M_\Theta : & H^2(\mathbb{D}) & \longrightarrow & H^2(\mathbb{D}) \\ & f & \longmapsto & \Theta f. \end{array}$$

Show that

$$\|\Theta f\|_{\mathcal{M}(M_\Theta)} = \|f\|_{H^2(\mathbb{D})} = \|\Theta f\|_{H^2(\mathbb{D})} \qquad (f \in H^2(\mathbb{D})).$$

Thus $\mathcal{M}(M_\Theta) = \Theta H^2$ is isometrically contained in $H^2(\mathbb{D})$.
Hint: M_Θ is injective and $|\Theta| = 1$ almost everywhere on $\mathbb{T}$.

Exercise 16.1.5 Let $A \in \mathcal{L}(H_1, H_2)$ and $\alpha \in \mathbb{C}$, $\alpha \neq 0$. Show that

$$\|w\|_{\mathcal{M}(\alpha A)} = \frac{\|w\|_{\mathcal{M}(A)}}{|\alpha|} \qquad (w \in \mathcal{M}(A)).$$

16.2 A characterization of $\mathcal{M}(A) \subset \mathcal{M}(B)$

If the operators $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$ are such that $\mathcal{M}(A) \Subset \mathcal{M}(B)$, then we surely have $\mathcal{M}(A) \subset \mathcal{M}(B)$. Conversely, if the set inclusion $\mathcal{M}(A) \subset \mathcal{M}(B)$ holds, then the inclusion mapping

$$\begin{array}{rccc} i : & \mathcal{M}(A) & \longrightarrow & \mathcal{M}(B) \\ & w & \longmapsto & w \end{array}$$

is well defined. But, in fact, more is true. The way that the structures of $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are defined forces i to be a bounded operator and thus $\mathcal{M}(A)$ is boundedly contained in $\mathcal{M}(B)$.

Lemma 16.6 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$ be such that $\mathcal{M}(A) \subset \mathcal{M}(B)$. Then $\mathcal{M}(A) \Subset \mathcal{M}(B)$.*

Proof We need to show that the inclusion $i : \mathcal{M}(A) \longrightarrow \mathcal{M}(B)$ is a bounded operator. The justification is based on the closed graph theorem. Let $(w_n)_{n\geq 1}$ be a sequence in $\mathcal{R}(A)$ that converges to w in $\mathcal{M}(A)$ and to w' in $\mathcal{M}(B)$. Note that $iw_n = w_n$. Since $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are both boundedly embedded into H, the sequence $(w_n)_{n\geq 1}$ also tends to w and to w' in the norm of H. Then, by uniqueness of the limit, we must have $w = w'$. Hence, the closed graph theorem implies that i is continuous. □

Lemma 16.6 shows that the new notation $\Subset$ is not needed in the study of $\mathcal{M}(A)$ spaces. However, we emphasize that $\mathcal{M}(A) = \mathcal{M}(B)$ is not equivalent to $\mathcal{M}(A) \doteq \mathcal{M}(B)$. The identity $\mathcal{M}(A) = \mathcal{M}(B)$ implies that

$$c\,\|w\|_{\mathcal{M}(B)} \leq \|w\|_{\mathcal{M}(A)} \leq C\,\|w\|_{\mathcal{M}(B)},$$

while in the definition of $\mathcal{M}(A) \doteq \mathcal{M}(B)$ we assumed that

$$\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(B)}.$$

To use Lemma 16.6, we naturally ask under what conditions the set inclusion $\mathcal{M}(A) \subset \mathcal{M}(B)$ holds. Let us treat a sufficient condition. Suppose that there is a bounded operator $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, with $\|C\| \leq c$, such that $A = BC$. Since, for each $x \in \mathcal{H}_1$, $Ax = B(Cx)$, we have the set inclusion $\mathcal{M}(A) \subset \mathcal{M}(B)$. Thus, by Lemma 16.6, $\mathcal{M}(A) \Subset \mathcal{M}(B)$. Moreover, by (16.7) and the fact that $\|C\| \leq c$, we have

$$\|Ax\|_{\mathcal{M}(B)} = \|BCx\|_{\mathcal{M}(B)} \leq \|Cx\|_{\mathcal{H}_2} \leq c\,\|x\|_{\mathcal{H}_1}.$$

By (16.5), replacing x by $P_{(\ker A)^\perp}x$ gives us

$$\|Ax\|_{\mathcal{M}(B)} \leq c\,\|Ax\|_{\mathcal{M}(A)} \qquad (x \in \mathcal{H}_1).$$

Hence, the norm of i is less than or equal to c. This means that $\mathcal{M}(A)$ is boundedly contained in $\mathcal{M}(B)$ and, in particular, if $c = 1$, $\mathcal{M}(A)$ is contractively contained in $\mathcal{M}(B)$. What is surprising is that the existence of C is also necessary for the bounded inclusion of $\mathcal{M}(A)$ in $\mathcal{M}(B)$.

Theorem 16.7 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$, and let $c > 0$. Then the following are equivalent.*

(i) $AA^* \leq c^2 BB^*$.
(ii) *There is an operator $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, with $\|C\| \leq c$, such that $A = BC$.*
(iii) *We have $\mathcal{M}(A) \subset \mathcal{M}(B)$ with*

$$\|w\|_{\mathcal{M}(B)} \leq c\,\|w\|_{\mathcal{M}(A)} \qquad (w \in \mathcal{M}(A)),$$

i.e. the inclusion $i : \mathcal{M}(A) \longrightarrow \mathcal{M}(B)$ is a bounded operator of norm less than or equal to c.

In particular, $\mathcal{M}(A) \hookrightarrow \mathcal{M}(B)$ if and only if $AA^* \leq BB^*$.

Proof (i) $\Longleftrightarrow$ (ii) This is the content of Theorem 7.11.

(ii) $\Longrightarrow$ (iii) This was discussed above.

(iii) $\Longrightarrow$ (ii) Take an element $w = Ax \in \mathcal{M}(A)$, with some $x \in \mathcal{H}_1$. Hence, for each $x \in \mathcal{H}_1$, there is a $y \in \mathcal{H}_2$ such that

$$Ax = By. \tag{16.14}$$

The element y is not necessarily unique. However, if $By = By'$, with $y, y' \in \mathcal{H}_2$, then $B(y - y') = 0$ and thus $y - y' \in \ker B$. In other words, we have $P_{(\ker B)^\perp} y = P_{(\ker B)^\perp} y'$. Therefore, the mapping

$$\begin{array}{rccl} C: & \mathcal{H}_1 & \longrightarrow & \mathcal{H}_2 \\ & x & \longmapsto & P_{(\ker B)^\perp} y, \end{array}$$

with $y \in \mathcal{H}_2$ given by (16.14), is well defined and

$$BCx = BP_{(\ker B)^\perp} y = By = Ax \qquad (x \in \mathcal{H}_1).$$

This means that the definition of C is adjusted such that the identity $A = BC$ holds. Moreover, by (16.5) and (16.7) and our assumption,

$$\begin{aligned} \|Cx\|_{\mathcal{H}_2} &= \|P_{(\ker B)^\perp} y\|_{\mathcal{H}_2} \\ &= \|By\|_{\mathcal{M}(B)} \\ &= \|Ax\|_{\mathcal{M}(B)} \\ &\le c\,\|Ax\|_{\mathcal{M}(A)} \\ &\le c\,\|x\|_{\mathcal{H}_1} \qquad (x \in \mathcal{H}_1). \end{aligned}$$

Hence, C is a bounded operator of norm less than or equal to c. □

We gather some important corollaries below. The first one follows immediately from Theorem 16.7.

Corollary 16.8 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H})$. Then the following statements hold.*

(i) $\mathcal{M}(A) \doteq \mathcal{M}(B)$ *if and only if* $AA^* = BB^*$.
(ii) $\mathcal{M}(A) \doteq \mathcal{M}(|A|)$, *where* $|A| = (AA^*)^{1/2}$.

If the linear manifold $\mathcal{R}(A)$ is closed in H, then it inherits the Hilbert space structure of H. One may wonder if this Hilbert space structure coincides with the one we put on $\mathcal{R}(A)$ and called it $\mathcal{M}(A)$. The following corollary answers this question.

Corollary 16.9 *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$. Then $\mathcal{R}(A)$ is a closed subspace of $\mathcal{H}$ and $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{H}}$, for each $w \in \mathcal{M}(A)$, if and only if A is a partial isometry. In this case, we have*

$$\mathcal{M}(A) \doteq \mathcal{M}(AA^*).$$