

Cambridge University Press

978-1-107-02777-0 - The Theory of $\mathcal{H}(b)$ Spaces: Volume 1

Emmanuel Fricain and Javad Mashreghi

Frontmatter

[More information](#)

The Theory of $\mathcal{H}(b)$ Spaces

Volume 1

An $\mathcal{H}(b)$ space is defined as a collection of analytic functions that are in the image of an operator. The theory of $\mathcal{H}(b)$ spaces bridges two classical subjects, complex analysis and operator theory, which makes it both appealing and demanding.

Volume 1 of this comprehensive treatment is devoted to the preliminary subjects required to understand the foundation of $\mathcal{H}(b)$ spaces, such as Hardy spaces, Fourier analysis, integral representation theorems, Carleson measures, Toeplitz and Hankel operators, various types of shift operators and Clark measures. Volume 2 focuses on the central theory. Both books are accessible to graduate students as well as researchers: each volume contains numerous exercises and hints, and figures are included throughout to illustrate the theory. Together, these two volumes provide everything the reader needs to understand and appreciate this beautiful branch of mathematics.

EMMANUEL FRICAIN is Professor of Mathematics at Laboratoire Paul Painlevé, Université Lille 1, France. Part of his research focuses on the interaction between complex analysis and operator theory, which is the main content of this book. He has a wealth of experience teaching numerous graduate courses on different aspects of analytic Hilbert spaces, and he has published several papers on $\mathcal{H}(b)$ spaces in high-quality journals, making him a world specialist in this subject.

JAVAD MASHREGHI is a Professor of Mathematics at Université Laval, Québec, Canada, where he has been selected Star Professor of the Year seven times for excellence in teaching. His main fields of interest are complex analysis, operator theory and harmonic analysis. He is the author of several mathematical textbooks, monographs and research articles. He won the G. de B. Robinson Award, the publication prize of the Canadian Mathematical Society, in 2004.

Cambridge University Press

978-1-107-02777-0 - The Theory of $\mathcal{H}(b)$ Spaces: Volume 1

Emmanuel Fricain and Javad Mashreghi

Frontmatter

[More information](#)

NEW MATHEMATICAL MONOGRAPHS

Editorial Board

Béla Bollobás, William Fulton, Anatole Katok, Frances Kirwan, Peter Sarnak, Barry Simon, Burt Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit www.cambridge.org/mathematics.

1. M. Cabanes and M. Enguehard *Representation Theory of Finite Reductive Groups*
2. J. B. Garnett and D. E. Marshall *Harmonic Measure*
3. P. Cohn *Free Ideal Rings and Localization in General Rings*
4. E. Bombieri and W. Gubler *Heights in Diophantine Geometry*
5. Y. J. Ionin and M. S. Shrikhande *Combinatorics of Symmetric Designs*
6. S. Berhanu, P. D. Cordaro and J. Hounie *An Introduction to Involutive Structures*
7. A. Shlapentokh *Hilbert's Tenth Problem*
8. G. Michler *Theory of Finite Simple Groups I*
9. A. Baker and G. Wüstholz *Logarithmic Forms and Diophantine Geometry*
10. P. Kronheimer and T. Mrowka *Monopoles and Three-Manifolds*
11. B. Bekka, P. de la Harpe and A. Valette *Kazhdan's Property (T)*
12. J. Neisendorfer *Algebraic Methods in Unstable Homotopy Theory*
13. M. Grandis *Directed Algebraic Topology*
14. G. Michler *Theory of Finite Simple Groups II*
15. R. Schertz *Complex Multiplication*
16. S. Bloch *Lectures on Algebraic Cycles (2nd Edition)*
17. B. Conrad, O. Gabber and G. Prasad *Pseudo-reductive Groups*
18. T. Downarowicz *Entropy in Dynamical Systems*
19. C. Simpson *Homotopy Theory of Higher Categories*
20. E. Fricain and J. Mashreghi *The Theory of $\mathcal{H}(b)$ Spaces I*
21. E. Fricain and J. Mashreghi *The Theory of $\mathcal{H}(b)$ Spaces II*
22. J. Goubault-Larrecq *Non-Hausdorff Topology and Domain Theory*
23. J. Śniatycki *Differential Geometry of Singular Spaces and Reduction of Symmetry*
24. E. Riehl *Categorical Homotopy Theory*
25. B. A. Munson and I. Volić *Cubical Homotopy Theory*
26. B. Conrad, O. Gabber and G. Prasad *Pseudo-reductive Groups (2nd Edition)*
27. J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson *Sobolev Spaces on Metric Measure Spaces*
28. Y.-G. Oh *Symplectic Topology and Floer Homology I*
29. Y.-G. Oh *Symplectic Topology and Floer Homology II*

Cambridge University Press
978-1-107-02777-0 - The Theory of $H(b)$ Spaces: Volume 1
Emmanuel Fricain and Javad Mashreghi
Frontmatter
[More information](#)

The Theory of $\mathcal{H}(b)$ Spaces

Volume 1

EMMANUEL FRICAIN
Université Lille 1

JAVAD MASHREGHI
Université Laval, Québec



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press
 978-1-107-02777-0 - The Theory of $H(b)$ Spaces: Volume 1
 Emmanuel Fricain and Javad Mashreghi
 Frontmatter
[More information](#)

CAMBRIDGE
 UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781107027770

© Emmanuel Fricain and Javad Mashreghi 2016

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2016

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

Fricain, Emmanuel, 1971 – author.

The theory of $H(b)$ spaces / Emmanuel Fricain, Javad Mashreghi.
 2 volumes ; cm. – (New mathematical monographs ; v. 20–21)

ISBN 978-1-107-02777-0 (Hardback)

1. Hilbert space. 2. Hardy spaces. 3. Analytic functions. 4. Linear operators. I. Mashreghi, Javad, author. II. Title.

QA322.4.F73 2014

515'.733–dc23 2014005539

ISBN – 2 Volume Set 978-1-107-11941-3 Hardback

ISBN – Volume 1 978-1-107-02777-0 Hardback

ISBN – Volume 2 978-1-107-02778-7 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Cambridge University Press

978-1-107-02777-0 - The Theory of $H(b)$ Spaces: Volume 1

Emmanuel Fricain and Javad Mashreghi

Frontmatter

[More information](#)

To our families:
Keiko & Shahzad
Hugo & Dorsa, Parisa, Golsa

Cambridge University Press

978-1-107-02777-0 - The Theory of $H(b)$ Spaces: Volume 1

Emmanuel Fricain and Javad Mashreghi

Frontmatter

[More information](#)

Contents for Volume 1

| | | |
|----------|--|------------------|
| | <i>Preface</i> | <i>page xvii</i> |
| 1 | Normed linear spaces and their operators | 1 |
| | 1.1 Banach spaces | 1 |
| | 1.2 Bounded operators | 9 |
| | 1.3 Fourier series | 14 |
| | 1.4 The Hahn–Banach theorem | 15 |
| | 1.5 The Baire category theorem and its consequences | 21 |
| | 1.6 The spectrum | 26 |
| | 1.7 Hilbert space and projections | 30 |
| | 1.8 The adjoint operator | 40 |
| | 1.9 Tensor product and algebraic direct sum | 45 |
| | 1.10 Invariant subspaces and cyclic vectors | 49 |
| | 1.11 Compressions and dilations | 52 |
| | 1.12 Angle between two subspaces | 54 |
| | Notes on Chapter 1 | 57 |
| 2 | Some families of operators | 60 |
| | 2.1 Finite-rank operators | 60 |
| | 2.2 Compact operators | 62 |
| | 2.3 Subdivisions of spectrum | 65 |
| | 2.4 Self-adjoint operators | 70 |
| | 2.5 Contractions | 77 |
| | 2.6 Normal and unitary operators | 78 |
| | 2.7 Forward and backward shift operators on ℓ^2 | 80 |
| | 2.8 The multiplication operator on $L^2(\mu)$ | 83 |
| | 2.9 Doubly infinite Toeplitz and Hankel matrices | 86 |
| | Notes on Chapter 2 | 92 |

| | | |
|----------|--|-----|
| 3 | Harmonic functions on the open unit disk | 96 |
| | 3.1 Nontangential boundary values | 96 |
| | 3.2 Angular derivatives | 98 |
| | 3.3 Some well-known facts in measure theory | 101 |
| | 3.4 Boundary behavior of $P\mu$ | 106 |
| | 3.5 Integral means of $P\mu$ | 110 |
| | 3.6 Boundary behavior of $Q\mu$ | 112 |
| | 3.7 Integral means of $Q\mu$ | 113 |
| | 3.8 Subharmonic functions | 116 |
| | 3.9 Some applications of Green's formula | 117 |
| | Notes on Chapter 3 | 120 |
| 4 | Hardy spaces | 122 |
| | 4.1 Hyperbolic geometry | 122 |
| | 4.2 Classic Hardy spaces H^p | 124 |
| | 4.3 The Riesz projection P_+ | 130 |
| | 4.4 Kernels of P_+ and P_- | 135 |
| | 4.5 Dual and predual of H^p spaces | 137 |
| | 4.6 The canonical factorization | 141 |
| | 4.7 The Schwarz reflection principle for H^1 functions | 148 |
| | 4.8 Properties of outer functions | 149 |
| | 4.9 A uniqueness theorem | 154 |
| | 4.10 More on the norm in H^p | 157 |
| | Notes on Chapter 4 | 163 |
| 5 | More function spaces | 166 |
| | 5.1 The Nevanlinna class \mathcal{N} | 166 |
| | 5.2 The spectrum of b | 171 |
| | 5.3 The disk algebra \mathcal{A} | 173 |
| | 5.4 The algebra $\mathcal{C}(\mathbb{T}) + H^\infty$ | 181 |
| | 5.5 Generalized Hardy spaces $H^p(\nu)$ | 183 |
| | 5.6 Carleson measures | 187 |
| | 5.7 Equivalent norms on H^2 | 198 |
| | 5.8 The corona problem | 202 |
| | Notes on Chapter 5 | 211 |
| 6 | Extreme and exposed points | 214 |
| | 6.1 Extreme points | 214 |
| | 6.2 Extreme points of $L^p(\mathbb{T})$ | 217 |
| | 6.3 Extreme points of H^p | 219 |
| | 6.4 Strict convexity | 224 |
| | 6.5 Exposed points of $\mathfrak{B}(\mathcal{X})$ | 227 |
| | 6.6 Strongly exposed points of $\mathfrak{B}(\mathcal{X})$ | 230 |
| | 6.7 Equivalence of rigidity and exposed points in H^1 | 232 |

| | | |
|----------|---|------------|
| 6.8 | Properties of rigid functions | 235 |
| 6.9 | Strongly exposed points of H^1 | 246 |
| | Notes on Chapter 6 | 254 |
| 7 | More advanced results in operator theory | 257 |
| 7.1 | The functional calculus for self-adjoint operators | 257 |
| 7.2 | The square root of a positive operator | 260 |
| 7.3 | Möbius transformations and the Julia operator | 269 |
| 7.4 | The Wold–Kolmogorov decomposition | 274 |
| 7.5 | Partial isometries and polar decomposition | 275 |
| 7.6 | Characterization of contractions on $\ell^2(\mathbb{Z})$ | 281 |
| 7.7 | Densely defined operators | 282 |
| 7.8 | Fredholm operators | 286 |
| 7.9 | Essential spectrum of block-diagonal operators | 291 |
| 7.10 | The dilation theory | 298 |
| 7.11 | The abstract commutant lifting theorem | 306 |
| | Notes on Chapter 7 | 310 |
| 8 | The shift operator | 314 |
| 8.1 | The bilateral forward shift operator Z_μ | 314 |
| 8.2 | The unilateral forward shift operator S | 321 |
| 8.3 | Commutants of Z and S | 328 |
| 8.4 | Cyclic vectors of S | 333 |
| 8.5 | When do we have $H^p(\mu) = L^p(\mu)$? | 336 |
| 8.6 | The unilateral forward shift operator S_μ | 342 |
| 8.7 | Reducing invariant subspaces of Z_μ | 351 |
| 8.8 | Simply invariant subspaces of Z_μ | 353 |
| 8.9 | Reducing invariant subspaces of S_μ | 360 |
| 8.10 | Simply invariant subspaces of S_μ | 361 |
| 8.11 | Cyclic vectors of Z_μ and S^* | 363 |
| | Notes on Chapter 8 | 372 |
| 9 | Analytic reproducing kernel Hilbert spaces | 376 |
| 9.1 | The reproducing kernel | 376 |
| 9.2 | Multipliers | 381 |
| 9.3 | The Banach algebra $\mathfrak{Mult}(\mathcal{H})$ | 383 |
| 9.4 | The weak kernel | 386 |
| 9.5 | The abstract forward shift operator $S_{\mathcal{H}}$ | 390 |
| 9.6 | The commutant of $S_{\mathcal{H}}$ | 392 |
| 9.7 | When do we have $\mathfrak{Mult}(\mathcal{H}) = H^\infty$? | 394 |
| 9.8 | Invariant subspaces of $S_{\mathcal{H}}$ | 396 |
| | Notes on Chapter 9 | 396 |

| | | |
|-----------|---|-----|
| 10 | Bases in Banach spaces | 399 |
| | 10.1 Minimal sequences | 399 |
| | 10.2 Schauder basis | 403 |
| | 10.3 The multipliers of a sequence | 411 |
| | 10.4 Symmetric, nonsymmetric and unconditional basis | 414 |
| | 10.5 Riesz basis | 422 |
| | 10.6 The mappings $J_{\mathfrak{X}}$, $V_{\mathfrak{X}}$ and $\Gamma_{\mathfrak{X}}$ | 425 |
| | 10.7 Characterization of the Riesz basis | 430 |
| | 10.8 Bessel sequences and the Feichtinger conjecture | 435 |
| | 10.9 Equivalence of Riesz and unconditional bases | 440 |
| | 10.10 Asymptotically orthonormal sequences | 442 |
| | Notes on Chapter 10 | 449 |
| 11 | Hankel operators | 454 |
| | 11.1 A matrix representation for H_{φ} | 454 |
| | 11.2 The norm of H_{φ} | 457 |
| | 11.3 Hilbert's inequality | 462 |
| | 11.4 The Nehari problem | 466 |
| | 11.5 More approximation problems | 470 |
| | 11.6 Finite-rank Hankel operators | 473 |
| | 11.7 Compact Hankel operators | 475 |
| | Notes on Chapter 11 | 478 |
| 12 | Toeplitz operators | 481 |
| | 12.1 The operator $T_{\varphi} \in \mathcal{L}(H^2)$ | 481 |
| | 12.2 Composition of two Toeplitz operators | 487 |
| | 12.3 The spectrum of T_{φ} | 490 |
| | 12.4 The kernel of T_{φ} | 494 |
| | 12.5 When is T_{φ} compact? | 499 |
| | 12.6 Characterization of rigid functions | 500 |
| | 12.7 Toeplitz operators on $H^2(\mu)$ | 503 |
| | 12.8 The Riesz projection on $L^2(\mu)$ | 506 |
| | 12.9 Characterization of invertibility | 511 |
| | 12.10 Fredholm Toeplitz operators | 515 |
| | 12.11 Characterization of surjectivity | 518 |
| | 12.12 The operator $X_{\mathcal{H}}$ and its invariant subspaces | 520 |
| | Notes on Chapter 12 | 522 |
| 13 | Cauchy transform and Clark measures | 526 |
| | 13.1 The space $\mathfrak{R}(\mathbb{D})$ | 526 |
| | 13.2 Boundary behavior of C_{μ} | 533 |
| | 13.3 The mapping K_{μ} | 534 |
| | 13.4 The operator $K_{\varphi} : L^2(\varphi) \rightarrow H^2$ | 541 |
| | 13.5 Functional calculus for S_{φ} | 545 |

Contents

xi

| | | |
|-----------|---|-----|
| 13.6 | Toeplitz operators with symbols in $L^2(\mathbb{T})$ | 551 |
| 13.7 | Clark measures μ_α | 555 |
| 13.8 | The Cauchy transform of μ_α | 562 |
| 13.9 | The function ρ | 563 |
| | Notes on Chapter 13 | 564 |
| 14 | Model subspaces K_Θ | 567 |
| 14.1 | The arithmetic of inner functions | 567 |
| 14.2 | A generator for K_Θ | 570 |
| 14.3 | The orthogonal projection P_Θ | 576 |
| 14.4 | The conjugation Ω_Θ | 579 |
| 14.5 | Minimal sequences of reproducing kernels in K_B | 580 |
| 14.6 | The operators J and M_Θ | 583 |
| 14.7 | Functional calculus for M_Θ | 589 |
| 14.8 | Spectrum of M_Θ and $\varphi(M_\Theta)$ | 593 |
| 14.9 | The commutant lifting theorem for M_Θ | 602 |
| 14.10 | Multipliers of K_Θ | 607 |
| | Notes on Chapter 14 | 608 |
| 15 | Bases of reproducing kernels and interpolation | 611 |
| 15.1 | Uniform minimality of $(k_{\lambda_n})_{n \geq 1}$ | 611 |
| 15.2 | The Carleson–Newman condition | 612 |
| 15.3 | Riesz basis of reproducing kernels | 618 |
| 15.4 | Nevanlinna–Pick interpolation problem | 621 |
| 15.5 | H^∞ -interpolating sequences | 623 |
| 15.6 | H^2 -interpolating sequences | 624 |
| 15.7 | Asymptotically orthonormal sequences | 627 |
| | Notes on Chapter 15 | 638 |
| | <i>References</i> | 641 |
| | <i>Symbol index</i> | 669 |
| | <i>Author index</i> | 673 |
| | <i>Subject index</i> | 677 |

Contents for Volume 2

Preface

- 16 The spaces $\mathcal{M}(A)$ and $\mathcal{H}(A)$**
- 16.1 The space $\mathcal{M}(A)$
 - 16.2 A characterization of $\mathcal{M}(A) \subset \mathcal{M}(B)$
 - 16.3 Linear functionals on $\mathcal{M}(A)$
 - 16.4 The complementary space $\mathcal{H}(A)$
 - 16.5 The relation between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$
 - 16.6 The overlapping space $\mathcal{M}(A) \cap \mathcal{H}(A)$
 - 16.7 The algebraic sum of $\mathcal{M}(A_1)$ and $\mathcal{M}(A_2)$
 - 16.8 A decomposition of $\mathcal{H}(A)$
 - 16.9 The geometric definition of $\mathcal{H}(A)$
 - 16.10 The Julia operator $\mathfrak{J}(A)$ and $\mathcal{H}(A)$
- Notes on Chapter 16
- 17 Hilbert spaces inside H^2**
- 17.1 The space $\mathcal{M}(u)$
 - 17.2 The space $\mathcal{M}(\bar{u})$
 - 17.3 The space $\mathcal{H}(b)$
 - 17.4 The space $\mathcal{H}(\bar{b})$
 - 17.5 Relations between different $\mathcal{H}(\bar{b})$ spaces
 - 17.6 $\mathcal{M}(\bar{u})$ is invariant under S and S^*
 - 17.7 Contractive inclusion of $\mathcal{M}(\varphi)$ in $\mathcal{M}(\bar{\varphi})$
 - 17.8 Similarity of S and $S_{\mathcal{H}}$
 - 17.9 Invariant subspaces of $Z_{\bar{u}}$ and $X_{\bar{u}}$
 - 17.10 An extension of Beurling's theorem
- Notes on Chapter 17

- 18 The structure of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$**
- 18.1 When is $\mathcal{H}(b)$ a closed subspace of H^2 ?
 - 18.2 When is $\mathcal{H}(b)$ a dense subset of H^2 ?
 - 18.3 Decomposition of $\mathcal{H}(b)$ spaces
 - 18.4 The reproducing kernel of $\mathcal{H}(b)$
 - 18.5 $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are invariant under $T_{\bar{\varphi}}$
 - 18.6 Some inhabitants of $\mathcal{H}(b)$
 - 18.7 The unilateral backward shift operators X_b and $X_{\bar{b}}$
 - 18.8 The inequality of difference quotients
 - 18.9 A characterization of membership in $\mathcal{H}(b)$
- Notes on Chapter 18
- 19 Geometric representation of $\mathcal{H}(b)$ spaces**
- 19.1 Abstract functional embedding
 - 19.2 A geometric representation of $\mathcal{H}(b)$
 - 19.3 A unitary operator from \mathbb{K}_b onto \mathbb{K}_{b^*}
 - 19.4 A contraction from $\mathcal{H}(b)$ to $\mathcal{H}(b^*)$
 - 19.5 Almost conformal invariance
 - 19.6 The Littlewood Subordination Theorem revisited
 - 19.7 The generalized Schwarz–Pick estimates
- Notes on Chapter 19
- 20 Representation theorems for $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$**
- 20.1 Integral representation of $\mathcal{H}(\bar{b})$
 - 20.2 \mathbf{K}_ρ intertwines S_ρ^* and $X_{\bar{b}}$
 - 20.3 Integral representation of $\mathcal{H}(b)$
 - 20.4 A contractive antilinear map on $\mathcal{H}(b)$
 - 20.5 Absolutely continuity of the Clark measure
 - 20.6 Inner divisors of the Cauchy transform
 - 20.7 V_b intertwines S_μ^* and X_b
 - 20.8 Analytic continuation of $\mathcal{H}(b)$ functions
 - 20.9 Multipliers of $\mathcal{H}(b)$
 - 20.10 Multipliers and Toeplitz operators
 - 20.11 Comparison of measures
- Notes on Chapter 20
- 21 Angular derivatives of $\mathcal{H}(b)$ functions**
- 21.1 Derivative in the sense of Carathéodory
 - 21.2 Angular derivatives and Clark measures
 - 21.3 Derivatives of Blaschke products
 - 21.4 Higher derivatives of b
 - 21.5 Approximating by Blaschke products
 - 21.6 Reproducing kernels for derivatives
 - 21.7 An interpolation problem

- 21.8 Derivatives of $\mathcal{H}(b)$ functions
 Notes on Chapter 21
- 22 Bernstein-type inequalities**
- 22.1 Passage between \mathbb{D} and \mathbb{C}_+
 22.2 Integral representations for derivatives
 22.3 The weight $w_{p,n}$
 22.4 Some auxiliary integral operators
 22.5 The operator $T_{p,n}$
 22.6 Distances to the level sets
 22.7 Carleson-type embedding theorems
 22.8 A formula of combinatorics
 22.9 Norm convergence for the reproducing kernels
 Notes on Chapter 22
- 23 $\mathcal{H}(b)$ spaces generated by a nonextreme symbol b**
- 23.1 The pair (a, b)
 23.2 Inclusion of $\mathcal{M}(u)$ into $\mathcal{H}(b)$
 23.3 The element f^+
 23.4 Analytic polynomials are dense in $\mathcal{H}(b)$
 23.5 A formula for $\|X_b f\|_b$
 23.6 Another representation of $\mathcal{H}(b)$
 23.7 A characterization of $\mathcal{H}(b)$
 23.8 More inhabitants of $\mathcal{H}(b)$
 23.9 Unbounded Toeplitz operators and $\mathcal{H}(b)$ spaces
 Notes on Chapter 23
- 24 Operators on $\mathcal{H}(b)$ spaces with b nonextreme**
- 24.1 The unilateral forward shift operator S_b
 24.2 A characterization of $H^\infty \subset \mathcal{H}(b)$
 24.3 Spectrum of X_b and X_b^*
 24.4 Comparison of measures
 24.5 The function F_λ
 24.6 The operator W_λ
 24.7 Invariant subspaces of $\mathcal{H}(b)$ under X_b
 24.8 Completeness of the family of difference quotients
 Notes on Chapter 24
- 25 $\mathcal{H}(b)$ spaces generated by an extreme symbol b**
- 25.1 A unitary map between $\mathcal{H}(\bar{b})$ and $L^2(\rho)$
 25.2 Analytic continuation of $f \in \mathcal{H}(\bar{b})$
 25.3 Analytic continuation of $f \in \mathcal{H}(b)$
 25.4 A formula for $\|X_b f\|_b$
 25.5 S^* -cyclic vectors in $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$
 25.6 Orthogonal decompositions of $\mathcal{H}(b)$

- 25.7 The closure of $\mathcal{H}(\bar{b})$ in $\mathcal{H}(b)$
 25.8 A characterization of $\mathcal{H}(b)$
 Notes on Chapter 25
- 26 Operators on $\mathcal{H}(b)$ spaces with b extreme**
 26.1 Spectrum of X_b and X_b^*
 26.2 Multipliers of $\mathcal{H}(b)$ spaces, extreme case, part I
 26.3 Comparison of measures
 26.4 Further characterizations of angular derivatives for b
 26.5 Model operator for Hilbert space contractions
 26.6 Conjugation and completeness of difference quotients
 Notes on Chapter 26
- 27 Inclusion between two $\mathcal{H}(b)$ spaces**
 27.1 A new geometric representation of $\mathcal{H}(b)$ spaces
 27.2 The class $\mathcal{I}nt(V_{b_1}, V_{b_2})$
 27.3 The class $\mathcal{I}nt(\mathcal{S}_{b_1}, \mathcal{S}_{b_2})$
 27.4 Relations between different $\mathcal{H}(b)$ spaces
 27.5 The rational case
 27.6 Coincidence between $\mathcal{H}(b)$ and $\mathcal{D}(\mu)$ spaces
 Notes on Chapter 27
- 28 Topics regarding inclusions $\mathcal{M}(a) \subset \mathcal{H}(\bar{b}) \subset \mathcal{H}(b)$**
 28.1 A sufficient and a necessary condition for $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$
 28.2 Characterizations of $\mathcal{H}(\bar{b}) = \mathcal{H}(b)$
 28.3 Multipliers of $\mathcal{H}(b)$, extreme case, part II
 28.4 Characterizations of $\mathcal{M}(a) = \mathcal{H}(b)$
 28.5 Invariant subspaces of S_b when $b(z) = (1+z)/2$
 28.6 Characterization of $\overline{\mathcal{M}(a)}^b = \mathcal{H}(b)$
 28.7 Characterization of the closedness of $\mathcal{M}(a)$ in $\mathcal{H}(b)$
 28.8 Boundary eigenvalues and eigenvectors of S_b^*
 28.9 The space $\mathcal{H}_0(b)$
 28.10 The spectrum of S_0
 Notes on Chapter 28
- 29 Rigid functions and strongly exposed points of H^1**
 29.1 Admissible and special pairs
 29.2 Rigid functions of H^1 and $\mathcal{H}(b)$ spaces
 29.3 Dimension of $\mathcal{H}_0(b)$
 29.4 S_b -invariant subspaces of $\mathcal{H}(b)$
 29.5 A necessary condition for nonrigidity
 29.6 Strongly exposed points and $\mathcal{H}(b)$ spaces
 Notes on Chapter 29

- 30 Nearly invariant subspaces and kernels of Toeplitz operators**
- 30.1 Nearly invariant subspaces and rigid functions
 - 30.2 The operator R_f
 - 30.3 Extremal functions
 - 30.4 A characterization of nearly invariant subspaces
 - 30.5 Description of kernels of Toeplitz operators
 - 30.6 A characterization of surjectivity for Toeplitz operators
 - 30.7 The right inverse of a Toeplitz operator
- Notes on Chapter 30
- 31 Geometric properties of sequences of reproducing kernels**
- 31.1 Completeness and minimality in $\mathcal{H}(b)$ spaces
 - 31.2 Spectral properties of rank one perturbation of X_b^*
 - 31.3 Orthonormal bases in $\mathcal{H}(b)$ spaces
 - 31.4 Riesz sequences of reproducing kernels in $\mathcal{H}(b)$
 - 31.5 The invertibility of distortion operator and Riesz bases
 - 31.6 Riesz sequences in $H^2(\mu)$ and in $\mathcal{H}(\bar{b})$
 - 31.7 Asymptotically orthonormal sequences and bases in $\mathcal{H}(b)$
 - 31.8 Stability of completeness and AOB
 - 31.9 Stability of Riesz bases
- Notes on Chapter 31

References

Symbol Index

Subject Index

Preface

In 1915, Godfrey Harold Hardy, in a famous paper published in the *Proceedings of the London Mathematical Society*, first put forward the “theory of Hardy spaces” H^p . Having a Hilbert space structure, H^2 also benefits from the rich theory of Hilbert spaces and their operators. The mutual interaction of analytic function theory, on the one hand, and operator theory, on the other, has created one of the most beautiful branches of mathematical analysis. The Hardy–Hilbert space H^2 is the glorious king of this seemingly small, but profoundly deep, territory.

In 1948, in the context of dynamics of Hilbert space operators, A. Beurling classified the closed invariant subspaces of the forward shift operator on ℓ^2 . The genuine idea of Beurling was to exploit the forward shift operator S on H^2 . To that end, he used some analytical tools to show that the closed subspaces of H^2 that are invariant under S are precisely of the form ΘH^2 , where Θ is an inner function. Therefore, the orthogonal complement of the Beurling subspace ΘH^2 , the so-called *model subspaces* K_Θ , are the closed invariant subspaces of H^2 that are invariant under the backward shift operator S^* . The model subspaces have rich algebraic and analytic structures with applications in other branches of mathematics and science, for example, control engineering and optics.

The word “model” that was used above to describe K_Θ refers to their application in recognizing the Hilbert space contractions. The main idea is to identify (via a unitary operator) a contraction as the adjoint of multiplication by z on a certain space of analytic functions on the unit disk. As Beurling’s theorem says, if we restrict ourselves to closed subspaces of H^2 that are invariant under S^* , we just obtain K_Θ spaces, where Θ runs through the family of inner functions. This point of view was exploited by B. Sz.-Nagy and C. Foiaş to construct a model for Hilbert space contractions. Another way is to consider submanifolds (not necessarily closed) of H^2 that are invariant under S^* . Above half a century ago, such a modeling theory was developed by L. de Branges and J. Rovnyak. In this context, they introduced the so-called $\mathcal{H}(b)$ spaces. The de Branges–Rovnyak model is, in a certain sense, more flexible, but it causes certain difficulties. For example, the inner product in $\mathcal{H}(b)$ is not given by an explicit integral formula, contrary to the case for K_Θ , which is actually the inner product of H^2 , and this makes the treatment of $\mathcal{H}(b)$ functions more difficult.

Cambridge University Press

978-1-107-02777-0 - The Theory of $\mathcal{H}(b)$ Spaces: Volume 1

Emmanuel Fricain and Javad Mashreghi

Frontmatter

[More information](#)

The original definition of $\mathcal{H}(b)$ spaces uses the notion of *complementary space*, which is a generalization of the orthogonal complement in a Hilbert space. But $\mathcal{H}(b)$ spaces can also be viewed as the range of a certain operator involving Toeplitz operators. This point of view was a turning point in the theory of $\mathcal{H}(b)$ spaces. Adopting the new definition, D. Sarason and several others made essential contributions to the theory. In fact, they now play a key role in many other questions of function theory (solution of the Bieberbach conjecture by de Branges, rigid functions of the unit ball of H^1 , Schwarz–Pick inequalities), operator theory (invariant subspaces problem, composition operators), system theory and control theory. An excellent but very concise account of the theory of $\mathcal{H}(b)$ spaces is available in Sarason’s masterpiece [460]. However, there are many results, both new and old, that are not covered there. On the other hand, despite many efforts, the structure and properties of $\mathcal{H}(b)$ spaces still remain mysterious, and numerous natural questions still remain open. However, these spaces have a beautiful structure, with numerous applications, and we hope that this work attracts more people to this domain.

In this context, we have tried to provide a rather comprehensive *introduction* to the theory of $\mathcal{H}(b)$ spaces. That is why Volume 1 is devoted to the foundation of $\mathcal{H}(b)$ spaces. In Volume 2, we discuss $\mathcal{H}(b)$ spaces and their applications. However, two facts should be kept in mind: first, we just treat the scalar case of $\mathcal{H}(b)$ spaces; and second, we do not discuss in detail the theory of model operators, because there are already two excellent monographs on this topic [388]; [508]. Nevertheless, some of the tools of model theory are implicitly exploited in certain topics. For instance, to treat some natural questions such as the inclusion between two different $\mathcal{H}(b)$ spaces, we use a geometric representation of $\mathcal{H}(b)$ spaces that comes from the relation between Sz.-Nagy–Foiiaş and de Branges–Rovnyak modeling theory. Also, even if the main point of view that has been adopted in this book is based on the definition of $\mathcal{H}(b)$ via Toeplitz operators, the historical definition of de Branges and Rovnyak is also discussed and used at some points.

In the past decade, both of us have made several transatlantic trips to meet each other and work together on this book project. For these visits, we have been financially supported by Université Claude Bernard Lyon I, Université Lille 1, Université Laval, McGill University, Centre Jacques-Cartier (France), CNRS (Centre National de la Recherche Scientifique, France), ANR (Agence Nationale de la Recherche), FQRNT (Fonds Québécois de la Recherche sur la Nature et les Technologies) and NSERC (Natural Sciences and Engineering Research Council of Canada). We also benefited from the warm hospitality of CIRM (Centre International des Rencontres Mathématiques, Luminy), CRM (Centre de Recherches Mathématiques, Montréal) and the Fields Institute (Toronto). We thank them all warmly for their support.

During these past years, parallel to the writing of this book, we have also pursued our research, and some projects were directly related to $\mathcal{H}(b)$ spaces. Some of the results, mainly in collaboration with other colleagues, are contained in this monograph. Hence, we would like to thank from the bottom of our hearts our close collaborators: A. Baranov, A. Blandignères, G. Chacon, I. Chalendar, N. Chevrot, F. Gaunard, A. Hartmann, W. Ross, M. Shabankhah and D. Timotin. During the preparation of the manuscript, we also benefited from very useful discussions with P. Gorkin and D. Timotin concerning certain points of this book. We would like to warmly thank them both.

The first author would like to thank T. Ransford for inviting him to Laval University in 2014. During his six-month visit, he met the second author and the seed of mutual collaboration on $\mathcal{H}(b)$ spaces was sown. He is also profoundly influenced in his mathematical life by N. Nikolskii. He takes this opportunity to express his profound gratitude and admiration.

The preparation of the manuscript was not an easy task, and it took several years even after acceptance for publication by Cambridge University Press. We thank the CUP team, who were ultra-patient with us and on numerous occasions intervened to settle difficulties. Above all, we are grateful to Roger Astley for his unique constructive leadership during the whole procedure.

Emmanuel Fricain
Lille

Javad Mashreghi
Kashan