# The Theory of $\mathcal{H}(b)$ Spaces

Volume 1

An $\mathcal{H}(b)$ space is defined as a collection of analytic functions that are in the image of an operator. The theory of $\mathcal{H}(b)$ spaces bridges two classical subjects, complex analysis and operator theory, which makes it both appealing and demanding.

Volume 1 of this comprehensive treatment is devoted to the preliminary subjects required to understand the foundation of $\mathcal{H}(b)$ spaces, such as Hardy spaces, Fourier analysis, integral representation theorems, Carleson measures, Toeplitz and Hankel operators, various types of shift operators and Clark measures. Volume 2 focuses on the central theory. Both books are accessible to graduate students as well as researchers: each volume contains numerous exercises and hints, and figures are included throughout to illustrate the theory. Together, these two volumes provide everything the reader needs to understand and appreciate this beautiful branch of mathematics.

Emmanuel Fricain is Professor of Mathematics at Laboratoire Paul Painlevé, Université Lille 1, France. Part of his research focuses on the interaction between complex analysis and operator theory, which is the main content of this book. He has a wealth of experience teaching numerous graduate courses on different aspects of analytic Hilbert spaces, and he has published several papers on $\mathcal{H}(b)$ spaces in high-quality journals, making him a world specialist in this subject.

Javad Mashreghi is a Professor of Mathematics at Université Laval, Québec, Canada, where he has been selected Star Professor of the Year seven times for excellence in teaching. His main fields of interest are complex analysis, operator theory and harmonic analysis. He is the author of several mathematical textbooks, monographs and research articles. He won the G. de B. Robinson Award, the publication prize of the Canadian Mathematical Society, in 2004.

NEW MATHEMATICAL MONOGRAPHS



All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit www.cambridge.org/mathematics.

1. M. Cabanes and M. Enguehard *Representation Theory of Finite Reductive Groups*
2. J. B. Garnett and D. E. Marshall *Harmonic Measure*
3. P. Cohn *Free Ideal Rings and Localization in General Rings*
4. E. Bombieri and W. Gubler *Heights in Diophantine Geometry*
5. Y. J. Ionin and M. S. Shrikhande *Combinatorics of Symmetric Designs*
6. S. Berhanu, P. D. Cordaro and J. Hounie *An Introduction to Involutive Structures*
7. A. Shlapentokh *Hilbert's Tenth Problem*
8. G. Michler *Theory of Finite Simple Groups I*
9. A. Baker and G. Wüstholz *Logarithmic Forms and Diophantine Geometry*
10. P. Kronheimer and T. Mrowka *Monopoles and Three-Manifolds*
11. B. Bekka, P. de la Harpe and A. Valette *Kazhdan's Property (T)*
12. J. Neisendorfer *Algebraic Methods in Unstable Homotopy Theory*
13. M. Grandis *Directed Algebraic Topology*
14. G. Michler *Theory of Finite Simple Groups II*
15. R. Schertz *Complex Multiplication*
16. S. Bloch *Lectures on Algebraic Cycles (2nd Edition)*
17. B. Conrad, O. Gabber and G. Prasad *Pseudo-reductive Groups*
18. T. Downarowicz *Entropy in Dynamical Systems*
19. C. Simpson *Homotopy Theory of Higher Categories*
20. E. Fricain and J. Mashreghi *The Theory of* $\mathcal{H}(b)$ *Spaces I*
21. E. Fricain and J. Mashreghi *The Theory of* $\mathcal{H}(b)$ *Spaces II*
22. J. Goubault-Larrecq *Non-Hausdorff Topology and Domain Theory*
23. J. Śniatycki *Differential Geometry of Singular Spaces and Reduction of Symmetry*
24. E. Riehl *Categorical Homotopy Theory*
25. B. A. Munson and I. Volić *Cubical Homotopy Theory*
26. B. Conrad, O. Gabber and G. Prasad *Pseudo-reductive Groups (2nd Edition)*
27. J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson *Sobolev Spaces on Metric Measure Spaces*
28. Y.-G. Oh *Symplectic Topology and Floer Homology I*
29. Y.-G. Oh *Symplectic Topology and Floer Homology II*

# The Theory of $\mathcal{H}(b)$ Spaces

## Volume 1

EMMANUEL FRICAIN
*Université Lille 1*

JAVAD MASHREGHI
*Université Laval, Québec*

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107027770



First published 2016

*A catalogue record for this publication is available from the British Library*

*Library of Congress Cataloguing in Publication data*
Fricain, Emmanuel, 1971 – author.
The theory of H(b) spaces / Emmanuel Fricain, Javad Mashreghi.
2 volumes ; cm. – (New mathematical monographs ; v. 20–21)
ISBN 978-1-107-02777-0 (Hardback)
1. Hilbert space. 2. Hardy spaces. 3. Analytic functions. 4. Linear operators. I. Mashreghi, Javad, author. II. Title.
QA322.4.F73 2014
515′.733–dc23 2014005539

ISBN – 2 Volume Set 978-1-107-11941-3 Hardback
ISBN – Volume 1 978-1-107-02777-0 Hardback
ISBN – Volume 2 978-1-107-02778-7 Hardback

To our families:
Keiko & Shahzad
Hugo & Dorsa, Parisa, Golsa

# Contents for Volume 1

# Contents for Volume 2

# Preface

In 1915, Godfrey Harold Hardy, in a famous paper published in the *Proceedings of the London Mathematical Society*, first put forward the "theory of Hardy spaces" $H^p$. Having a Hilbert space structure, $H^2$ also benefits from the rich theory of Hilbert spaces and their operators. The mutual interaction of analytic function theory, on the one hand, and operator theory, on the other, has created one of the most beautiful branches of mathematical analysis. The Hardy–Hilbert space $H^2$ is the glorious king of this seemingly small, but profoundly deep, territory.

In 1948, in the context of dynamics of Hilbert space operators, A. Beurling classified the closed invariant subspaces of the forward shift operator on $\ell^2$. The genuine idea of Beurling was to exploit the forward shift operator $S$ on $H^2$. To that end, he used some analytical tools to show that the closed subspaces of $H^2$ that are invariant under $S$ are precisely of the form $\Theta H^2$, where $\Theta$ is an inner function. Therefore, the orthogonal complement of the Beurling subspace $\Theta H^2$, the so-called *model subspaces* $K_\Theta$, are the closed invariant subspaces of $H^2$ that are invariant under the backward shift operator $S^*$. The model subspaces have rich algebraic and analytic structures with applications in other branches of mathematics and science, for example, control engineering and optics.

The word "model" that was used above to describe $K_\Theta$ refers to their application in recognizing the Hilbert space contractions. The main idea is to identify (via a unitary operator) a contraction as the adjoint of multiplication by $z$ on a certain space of analytic functions on the unit disk. As Beurling's theorem says, if we restrict ourselves to closed subspaces of $H^2$ that are invariant under $S^*$, we just obtain $K_\Theta$ spaces, where $\Theta$ runs through the family of inner functions. This point of view was exploited by B. Sz.-Nagy and C. Foiaş to construct a model for Hilbert space contractions. Another way is to consider submanifolds (not necessarily closed) of $H^2$ that are invariant under $S^*$. Above half a century ago, such a modeling theory was developed by L. de Branges and J. Rovnyak. In this context, they introduced the so-called $\mathcal{H}(b)$ spaces. The de Branges–Rovnyak model is, in a certain sense, more flexible, but it causes certain difficulties. For example, the inner product in $\mathcal{H}(b)$ is not given by an explicit integral formula, contrary to the case for $K_\Theta$, which is actually the inner product of $H^2$, and this makes the treatment of $\mathcal{H}(b)$ functions more difficult.

The original definition of $\mathcal{H}(b)$ spaces uses the notion of *complementary space*, which is a generalization of the orthogonal complement in a Hilbert space. But $\mathcal{H}(b)$ spaces can also be viewed as the range of a certain operator involving Toeplitz operators. This point of view was a turning point in the theory of $\mathcal{H}(b)$ spaces. Adopting the new definition, D. Sarason and several others made essential contributions to the theory. In fact, they now play a key role in many other questions of function theory (solution of the Bieberbach conjecture by de Branges, rigid functions of the unit ball of $H^1$, Schwarz–Pick inequalities), operator theory (invariant subspaces problem, composition operators), system theory and control theory. An excellent but very concise account of the theory of $\mathcal{H}(b)$ spaces is available in Sarason's masterpiece [460]. However, there are many results, both new and old, that are not covered there. On the other hand, despite many efforts, the structure and properties of $\mathcal{H}(b)$ spaces still remain mysterious, and numerous natural questions still remain open. However, these spaces have a beautiful structure, with numerous applications, and we hope that this work attracts more people to this domain.

In this context, we have tried to provide a rather comprehensive *introduction* to the theory of $\mathcal{H}(b)$ spaces. That is why Volume 1 is devoted to the foundation of $\mathcal{H}(b)$ spaces. In Volume 2, we discuss $\mathcal{H}(b)$ spaces and their applications. However, two facts should be kept in mind: first, we just treat the scalar case of $\mathcal{H}(b)$ spaces; and second, we do not discuss in detail the theory of model operators, because there are already two excellent monographs on this topic [388]; [508]. Nevertheless, some of the tools of model theory are implicitly exploited in certain topics. For instance, to treat some natural questions such as the inclusion between two different $\mathcal{H}(b)$ spaces, we use a geometric representation of $\mathcal{H}(b)$ spaces that comes from the relation between Sz.-Nagy–Foiaş and de Branges–Rovnyak modeling theory. Also, even if the main point of view that has been adopted in this book is based on the definition of $\mathcal{H}(b)$ via Toeplitz operators, the historical definition of de Branges and Rovnyak is also discussed and used at some points.

In the past decade, both of us have made several transatlantic trips to meet each other and work together on this book project. For these visits, we have been financially supported by Université Claude Bernard Lyon I, Université Lille 1, Université Laval, McGill University, Centre Jacques-Cartier (France), CNRS (Centre National de la Recherche Scientifique, France), ANR (Agence Nationale de la Recherche), FQRNT (Fonds Québécois de la Recherche sur la Nature et les Technologies) and NSERC (Natural Sciences and Engineering Research Council of Canada). We also benefited from the warm hospitality of CIRM (Centre International des Rencontres Mathématiques, Luminy), CRM (Centre de Recherches Mathématiques, Montréal) and the Fields Institute (Toronto). We thank them all warmly for their support.

During these past years, parallel to the writing of this book, we have also pursued our research, and some projects were directly related to $\mathcal{H}(b)$ spaces. Some of the results, mainly in collaboration with other colleagues, are contained in this monograph. Hence, we would like to thank from the bottom of our hearts our close collaborators: A. Baranov, A. Blandignères, G. Chacon, I. Chalendar, N. Chevrot, F. Gaunard, A. Hartmann, W. Ross, M. Shabankhah and D. Timotin. During the preparation of the manuscript, we also benefited from very useful discussions with P. Gorkin and D. Timotin concerning certain points of this book. We would like to warmly thank them both.

The first author would like to thank T. Ransford for inviting him to Laval University in 2014. During his six-month visit, he met the second author and the seed of mutual collaboration on $\mathcal{H}(b)$ spaces was sown. He is also profoundly influenced in his mathematical life by N. Nikolskii. He takes this opportunity to express his profound gratitude and admiration.

The preparation of the manuscript was not an easy task, and it took several years even after acceptance for publication by Cambridge University Press. We thank the CUP team, who were ultra-patient with us and on numerous occasions intervened to settle difficulties. Above all, we are grateful to Roger Astley for his unique constructive leadership during the whole procedure.

Emmanuel Fricain — Javad Mashreghi
Lille — Kashan