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Normed linear spaces and their operators

In this chapter, we gather some basic facts about complex normed linear spaces and their operators. In particular, we discuss Banach spaces, Hilbert spaces and their bounded operators. There is no doubt that the subject is very vast and it is impossible to give a comprehensive treatment in one chapter. Our goal is to recall a few important aspects of the theory that are used in the study of $\mathcal{H}(b)$ spaces. We start by giving some examples of Banach spaces and introduce some classic operators. Then the dual space is defined and the well-known Hahn–Banach theorem is stated without proof. However, some applications of this essential result are outlined. Then we discuss the open mapping theorem (Theorem 1.14), the inverse mapping theorem (Corollary 1.15), the closed graph theorem (Corollary 1.18) and the uniform boundedness principle (Theorem 1.19). The common root of each of these theorems stems from the Baire category theorem (Theorem 1.13). Then we discuss Banach algebras and introduce the important concept of spectrum and state a simple version of the spectral mapping theorem (Theorem 1.22). At the end, we focus on Hilbert spaces, and some of their essential properties are outlined. We talk about Parseval's identity, the generalized version of the polarization identity, and Bessel's inequality. We also discuss in detail the compression of an operator to a closed subspace. Then we consider several topologies that one may face on a Hilbert space or on the space of its operators. The important concepts of adjoint and tensor product are discussed next. The chapter ends with some elementary facts about invariant subspaces and the cyclic vectors.

1.1 Banach spaces

Throughout this text we will consider only complex normed linear spaces. A complete normed linear space is called a *Banach space*. The term *linear manifold* refers to subsets of a linear space that are closed under the algebraic operations, while the term *subspace* is reserved for linear manifolds that are also closed in the norm or metric topology. Nevertheless, the combination *closed subspace* can also be found in the text. Given a subset $\mathcal{E}$ of a Banach space $\mathcal{X}$, we denote by $\text{Lin}(\mathcal{E})$ the linear manifold spanned by $\mathcal{E}$, which is the linear space whose elements are finite linear combinations of elements of $\mathcal{E}$. The closure of $\text{Lin}(\mathcal{E})$ in $\mathcal{X}$ is denoted by $\text{Span}_{\mathcal{X}}(\mathcal{E})$, or by $\text{Span}(\mathcal{E})$ if there is no ambiguity.

We start by briefly discussing some relevant families of Banach spaces that we encounter in the sequel. For a sequence $\mathfrak{z} = (z_n)_{n\geq 1}$ of complex numbers, define

$$\|\mathfrak{z}\|_p = \left(\sum_{n=1}^{\infty} |z_n|^p \right)^{1/p} \qquad (0 < p < \infty)$$

and

$$\|\mathfrak{z}\|_\infty = \sup_{n\geq 1} |z_n|.$$

Then the sequence space $\ell^p = \ell^p(\mathbb{N})$, $0 < p \leq \infty$, consists of all sequences $\mathfrak{z}$ with $\|\mathfrak{z}\|_p < \infty$. Addition and scalar multiplication are defined componentwise in ℓ^p. With this setting, $(\ell^p, \|\cdot\|_p)$, $1 \leq p \leq \infty$, is a Banach space. The sequence space

$$c_0 = c_0(\mathbb{N}) = \{\mathfrak{z} : \lim_{n\to\infty} z_n = 0\}$$

is a closed subspace of ℓ^∞. In certain applications, it is more appropriate to let the index parameter n start from zero or from $-\infty$. In the latter case, we denote our spaces by $\ell^p(\mathbb{Z})$ and $c_0(\mathbb{Z})$.

For each n, let $\mathfrak{e}_n$ be the sequence whose components are all equal to 0 except in the nth place, which is equal to 1. Clearly each $\mathfrak{e}_n$ belongs to all sequence spaces introduced above. These elements will repeatedly enter our discussion.

Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu \geq 0$. For a measurable function f, let

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p} \qquad (0 < p < \infty)$$

and let

$$\|f\|_\infty = \inf\{\, M : |f(x)| \leq M,\ x \in X \setminus E,\ E \in \mathcal{A},\ \mu(E) = 0 \,\}.$$

Then the family of Lebesgue spaces

$$L^p(X,\mu) = L^p(X) = L^p(\mu) = \{f : \|f\|_p < \infty\}$$

is another important example that we will need. To emphasize the role of μ, we sometimes use the more detailed notation $\|f\|_{L^p(\mu)}$ for $\|f\|_p$. For $1 \leq p \leq \infty$, $(L^p(X,\mu), \|\cdot\|_p)$ is a Banach space. In fact, the sequence spaces $\ell^p(\mathbb{N})$ can be considered as a special family in this category. It is enough to let $\mathcal{A} = \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$, and consider the counting measure

$$\mu(E) = \begin{cases} n, & \text{if } E \text{ is finite and has } n \text{ elements,} \\ \infty, & \text{if } E \text{ is infinite,} \end{cases}$$

on $\mathbb{N}$. Then we have $\ell^p(\mathbb{N}) = L^p(\mathbb{N}, \mu)$.

Two more special classes are vital in practice. The unit circle $\mathbb{T}$ equipped with the normalized Lebesgue measure

$$dm(e^{it}) = \frac{dt}{2\pi}$$

and the real line $\mathbb{R}$ equipped with the Lebesgue measure dt give rise to the classic Lebesgue spaces $L^p(\mathbb{T})$ and $L^p(\mathbb{R})$. If E is a Borel subset of $\mathbb{T}$, then $|E|$ will denote its length with respect to the normalized Lebesgue measure, that is

$$|E| = m(E).$$

The Banach space $(\ell^\infty, \|\cdot\|_\infty)$ is equipped with a third operation. Besides vector addition and scalar multiplication, we have vector multiplication in this space. Given $\mathfrak{x} = (x_n)_{n\geq 1}$ and $\mathfrak{y} = (y_n)_{n\geq 1}$ in ℓ^∞, let

$$\mathfrak{x}\,\mathfrak{y} = (x_n y_n)_{n\geq 1}.$$

Clearly $\mathfrak{x}\,\mathfrak{y} \in \ell^\infty$, and ℓ^∞ with this operation is an algebra that satisfies

$$\|\mathfrak{x}\,\mathfrak{y}\|_\infty \leq \|\mathfrak{x}\|_\infty \, \|\mathfrak{y}\|_\infty.$$

If a Banach space $\mathcal{B}$ is equipped with a multiplication operation that turns it into an algebra, then it is called a *Banach algebra* if it satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \, \|y\| \qquad (x \in \mathcal{B},\ y \in \mathcal{B}). \tag{1.1}$$

If, furthermore, the multiplication has a unit element, i.e. a vector $\mathfrak{e}$ such that

$$x\,\mathfrak{e} = \mathfrak{e}\,x = x \qquad (x \in \mathcal{B})$$

and

$$\|\mathfrak{e}\| = 1,$$

then it is called a *unital Banach algebra*. The sequence space ℓ^∞ is our first example of a unital commutative Banach algebra with the unit element

$$\mathfrak{e} = (1, 1, 1, \dots).$$

Another example of a unital commutative Banach algebra is $\mathcal{C}(\mathbb{T})$, the family of continuous functions on $\mathbb{T}$ endowed with the norm

$$\|f\|_\infty = \max_{\zeta\in\mathbb{T}} |f(\zeta)|.$$

The unit element is the constant function 1. The monomials

$$\chi_n(\zeta) = \zeta^n \qquad (n \in \mathbb{Z},\ \zeta \in \mathbb{T})$$

are clearly in $\mathcal{C}(\mathbb{T})$. With this notation, the unit element is χ_0. However, for simplicity, we usually write z^n for χ_n. In particular, we denote the constant function χ_0 by 1.

A linear combination of χ_n, $n \in \mathbb{Z}$, is called a *trigonometric polynomial*, and the linear manifold of all trigonometric polynomials is denoted by $\mathcal{P}$. However, if we restrict n to be a nonnegative integer, then a linear combination of χ_n, $n \geq 0$, is called an *analytic polynomial*, and the linear manifold of all analytic polynomials is denoted by $\mathcal{P}_+$. The term "analytic" comes from the fact that each element of $\mathcal{P}_+$ extends to an analytic function on the complex plane. Similarly, $\mathcal{P}_-$ denotes the linear manifold created by χ_n, $n \leq -1$, and $\mathcal{P}_{0+}$ denotes the linear manifold created by χ_n, $n \geq 1$. In other words, $\mathcal{P}_{0+}$ is the linear manifold of analytic polynomials vanishing at 0.

The family of all complex Borel measures on the unit circle $\mathbb{T}$ is denoted by $\mathcal{M}(\mathbb{T})$. The set of positive measures in $\mathcal{M}(\mathbb{T})$ is denoted by $\mathcal{M}^+(\mathbb{T})$. Recall that all measures in $\mathcal{M}(\mathbb{T})$ are necessarily finite. The normalized *Lebesgue measure* m is a distinguished member of this class. Another interesting example is the *Dirac measure* δ_α, which attributes a unit mass to the point $\alpha \in \mathbb{T}$. For each $\mu \in \mathcal{M}(\mathbb{T})$, the smallest positive Borel measure ν that satisfies the inequality

$$|\mu(E)| \leq \nu(E)$$

for all Borel sets $E \subset \mathbb{T}$ is called the *total variation measure* of μ and is denoted by $|\mu|$. The total variation $|\mu|$ is also given by the formula

$$|\mu|(E) = \sup \sum_{k=1}^{n} |\mu(E_k)|,$$

where the supremum is taken over all possible partitions $\{E_1, E_2, \dots, E_n\}$ of E by Borel sets. The norm of an element $\mu \in \mathcal{M}(\mathbb{T})$ is defined to be its total variation on $\mathbb{T}$, i.e.

$$\|\mu\| = |\mu|(\mathbb{T}).$$

Then $\mathcal{M}(\mathbb{T})$, endowed with the above norm, is a Banach space. However, with an appropriate definition of product (the convolution of two measures) on $\mathcal{M}(\mathbb{T})$, this space becomes a commutative unital Banach algebra. But, for our applications, we do not need to treat the details of this operation. A measure $\mu \in \mathcal{M}(\mathbb{T})$ that takes only real values is called a *signed measure*. Note that we assume in the definition of signed measure that they are finite.

The nth Fourier coefficient of a measure $\mu \in \mathcal{M}(\mathbb{T})$ is defined by

$$\hat{\mu}(n) = \int_{\mathbb{T}} \chi_{-n}\, d\mu = \int_{\mathbb{T}} e^{-int}\, d\mu(e^{it}) \qquad (n \in \mathbb{Z}).$$

In the particular case $d\mu = \varphi\, dm$, where $\varphi \in L^1(\mathbb{T})$, we use $\hat{\varphi}(n)$ to denote the nth Fourier coefficient of μ, that is

$$\hat{\varphi}(n) = \int_{\mathbb{T}} \varphi \chi_{-n}\, dm = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it})\, e^{-int}\, dt \qquad (n \in \mathbb{Z}).$$

We refer to the sequence $(\hat{\varphi}(n))_{n\in\mathbb{Z}}$ as the *spectrum* of φ. For example, the negative part of the spectrum of any analytic polynomials is identically zero. The *uniqueness theorem* for the Fourier coefficients says that $\mu = 0$ if and only if $\hat{\mu}(n) = 0$, $n \in \mathbb{Z}$.

There is a method to connect an arbitrary Lebesgue integral to a standard Riemann integral on $\mathbb{R}$. To do so, for a measurable function f defined on a measure space (X, μ), we define the *distribution function*

$$m_{\mu,f}(t) = \mu(\{x \in X : |f(x)| > t\}).$$

For simplicity, instead of $m_{\mu,f}$, we sometimes write m_μ or m_f, or even m whenever there is no confusion. The promised connection is described in the following result.

Lemma 1.1 *Let (X, μ) be a measure space, and let $f : X \longrightarrow \mathbb{C}$ be a measurable function. Then, for any $0 < p < \infty$, we have*

$$\int_X |f|^p\, d\mu = \int_0^\infty p\, t^{p-1} m_{\mu,f}(t)\, dt.$$

Proof For $x \in X$ and $t > 0$, define

$$A_{x,t} = \{x \in X : |f(x)| > t\}$$

and let $\chi_{A_{x,t}}$ be the characteristic function of the set $A_{x,t}$. Then, by Fubini's theorem, we have

$$\begin{aligned}
\int_0^\infty p\, t^{p-1} m(t)\, dt &= \int_0^\infty p\, t^{p-1} \mu(\{x \in X : |f(x)| > t\})\, dt \\
&= \int_0^\infty p\, t^{p-1} \left(\int_X \chi_{A_{x,t}}(x)\, d\mu(x) \right) dt \\
&= \int_X \left(\int_0^{|f(x)|} p\, t^{p-1}\, dt \right) d\mu(x) \\
&= \int_X |f(x)|^p\, d\mu(x).
\end{aligned}$$

This completes the proof. □

As a special case of Lemma 1.1, we have the interesting formula

$$\int_X |f| \, d\mu = \int_0^\infty m_{\mu,f}(t) \, dt.$$

We will also need the following elementary result about convergence in $L^1(\mathbb{T})$.

Lemma 1.2 *Let $f \in L^1(\mathbb{T})$ and $(f_n)_{n\geq 1}$ be a sequence of nonnegative functions in $L^1(\mathbb{T})$ such that $\|f_n\|_1 = \|f\|_1$, $n \geq 1$, and $f_n \longrightarrow f$ a.e. on $\mathbb{T}$. Then f_n tends to f in $L^1(\mathbb{T})$, i.e.*

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0.$$

Proof Define the set $E_n = \{\zeta \in \mathbb{T} : f_n(\zeta) > f(\zeta)\}$. Since $\|f_n\|_1 = \|f\|_1$, we have

$$\int_{E_n} (f_n - f) \, dm = \int_{\mathbb{T}\setminus E_n} (f - f_n) \, dm.$$

This implies that

$$\begin{aligned} \|f_n - f\|_1 &= \int_{\mathbb{T}} |f_n - f| \, dm \\ &= \int_{E_n} (f_n - f) \, dm + \int_{\mathbb{T}\setminus E_n} (f - f_n) \, dm \\ &= 2 \int_{\mathbb{T}\setminus E_n} (f - f_n) \, dm. \end{aligned}$$

But $f_n - f \longrightarrow 0$ and $0 \leq f - f_n \leq f$ a.e. on $\mathbb{T} \setminus E_n$. Hence, the dominated Lebesgue convergence theorem implies that

$$\int_{\mathbb{T}\setminus E_n} (f - f_n) \, dm \longrightarrow 0 \qquad (n \longrightarrow \infty),$$

which gives the result. □

Exercises

Exercise 1.1.1 Show that, for each $0 < p < 1$, $\|\cdot\|_p$ does not fulfill the triangle inequality.
Remark: That is why ℓ^p, or more generally $L^p(X, \mu)$, $0 < p < 1$, is not a Banach space.

Exercise 1.1.2 Let $p < \infty$. Show that the linear manifold generated by $\mathfrak{e}_n$, $n \ge 1$, is dense in ℓ^p. What is the closure of this manifold in ℓ^∞?

Exercise 1.1.3 Show that ℓ^p, $0 < p < \infty$, is *separable*.
Hint: Use Exercise 1.1.2.
Remark: We recall that a space is called "separable" if it has a countable dense subset.

Exercise 1.1.4 Show that ℓ^∞ is not separable.
Hint: What is $\|\mathfrak{e}_m - \mathfrak{e}_n\|_\infty$?

Exercise 1.1.5 Consider the spaces of two-sided sequences,

$$\ell^p(\mathbb{Z}) = \left\{ (z_n)_{n\in\mathbb{Z}} : \|(z_n)_{n\in\mathbb{Z}}\|_p^p = \sum_{n=-\infty}^{\infty} |z_n|^p < \infty \right\}$$

and

$$\ell^\infty(\mathbb{Z}) = \left\{ (z_n)_{n\in\mathbb{Z}} : \|(z_n)_{n\in\mathbb{Z}}\|_\infty = \sup_{n\in\mathbb{Z}} |z_n| < \infty \right\}.$$

Let $\mathfrak{x} = (x_n)_{n\in\mathbb{Z}}$ and $\mathfrak{y} = (y_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. Define $\mathfrak{x} * \mathfrak{y} = (z_n)_{n\in\mathbb{Z}}$, where

$$z_n = \sum_{m=-\infty}^{\infty} x_m\, y_{n-m} \qquad (n \in \mathbb{Z}).$$

Show that the operation $*$ is well defined on $\ell^1(\mathbb{Z})$ and, moreover, that $\ell^1(\mathbb{Z})$ equipped with $*$ is a unital commutative Banach algebra.
Remark: The operation $*$ is called *convolution*.
Hint: We have

$$\sum_{n=1}^{\infty} \left(\sum_{m=-\infty}^{\infty} |x_m\, y_{n-m}| \right) = \left(\sum_{m=-\infty}^{\infty} |x_m| \right) \left(\sum_{k=-\infty}^{\infty} |y_k| \right).$$

Exercise 1.1.6 Let

$$\ell^p(\mathbb{Z}^+) = \{ (z_n)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}) : z_n = 0,\ n \le -1 \}.$$

Show that $\ell^p(\mathbb{Z}^+)$, $0 < p \le \infty$, is closed in $\ell^p(\mathbb{Z})$.

Exercise 1.1.7 The family of sequences of compact support is defined by

$$c_{00} = \{(z_n)_{n\ge1} : \exists\, N \text{ such that } z_n = 0,\ n \ge N\}.$$

Show that c_{00} is dense in c_0.

Exercise 1.1.8 Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu \geq 0$. Show that

$$\|f\|_\infty = \inf\{M : \mu(\{x \in X : |f(x)| > M\}) = 0\}.$$

Exercise 1.1.9 Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that $L^\infty(X)$ endowed with the pointwise multiplication is a unital commutative Banach algebra.

Exercise 1.1.10 Show that $L^1(\mathbb{R})$ equipped with the convolution operation

$$(f * g)(x) = \int_{\mathbb{R}} f(t)\, g(x-t)\, dt$$

is a nonunital commutative Banach algebra. Why does $L^1(\mathbb{R})$ not have a unit?

Exercise 1.1.11 Show that $L^1(\mathbb{T})$ equipped with the convolution operation

$$(f * g)(e^{i\theta}) = \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{it})\, g(e^{i(\theta-t)})\, dt$$

is a nonunital commutative Banach algebra. Why does $L^1(\mathbb{T})$ not have a unit?

Exercise 1.1.12 Let $(b_k)_{k\geq 1}$ be a decreasing sequence of nonnegative real numbers such that the series $\sum_{k=1}^{\infty} b_k$ converges.

(i) Using Abel's summation technique, show that

$$\sum_{k=0}^{N} k(b_k - b_{k+1}) \leq \sum_{k=1}^{N} b_k.$$

Deduce that the series $\sum_{k=0}^{N} k(b_k - b_{k+1})$ is convergent.

(ii) Show that the series $\sum_{k=1}^{\infty} (b_k - b_{k+1})$ is convergent and

$$b_n = \sum_{k=n}^{\infty} (b_k - b_{k+1}).$$

(iii) Deduce that

$$0 \leq n b_n \leq \sum_{k=n}^{\infty} k(b_k - b_{k+1}).$$

(iv) Conclude that $n b_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Exercise 1.1.13 Let $\mathcal{X}$ be a Banach space, let E be a linear manifold of X and let $x \in \mathcal{X}$. Show that $x \in \text{Clos}(E)$ if and only if there exists a sequence $(x_n)_n$ of vectors of E such that $\|x_n\| \leq 1/2^n$, $n \geq 2$ and

$$x = \sum_{n=1}^{\infty} x_n.$$

Hint: If $x = \lim_{n\to\infty} u_n$ with $u_n \in E$, then consider a subsequence $(u_{\varphi(n)})_n$ such that

$$\|u_{\varphi(n+1)} - u_{\varphi(n)}\| \leq \frac{1}{2^n} \qquad (n \geq 1).$$

Then, put $x_1 = u_{\varphi(1)}$ and $x_n = u_{\varphi(n)} - u_{\varphi(n-1)}$, $n \geq 2$.

1.2 Bounded operators

Let $\mathcal{X}$, $\mathcal{X}_1$ and $\mathcal{X}_2$ be normed linear spaces. The family of all *linear and continuous maps* from $\mathcal{X}_1$ into $\mathcal{X}_2$ is denoted by $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$, and we will write $\mathcal{L}(\mathcal{X})$ for $\mathcal{L}(\mathcal{X}, \mathcal{X})$. Given a linear map $A : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$, it is a well-known result that A is continuous if and only if

$$\|A\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)} = \sup_{\substack{x\in\mathcal{X}_1 \\ x\neq 0}} \frac{\|Ax\|_{\mathcal{X}_2}}{\|x\|_{\mathcal{X}_1}} < \infty. \tag{1.2}$$

If there is no ambiguity, we will also write $\|A\|$ for $\|A\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)}$. A linear map A belongs to $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ if and only if it satisfies

$$\|Ax\|_{\mathcal{X}_2} \leq C\|x\|_{\mathcal{X}_1} \qquad (x \in \mathcal{X}_1) \tag{1.3}$$

and $\|A\|$ is the infimum of C satisfying (1.3). That is why the elements of $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ are called *bounded operators*. If $\mathcal{X}_2$ is a Banach space, then the space $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ endowed with the norm $\|\cdot\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)}$ is also a Banach space. In the special case where $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$ is a Banach space, the space $\mathcal{L}(\mathcal{X})$, equipped with the composition of operators as its multiplication, is a unital noncommutative Banach algebra.

In the definition (1.2), the supremum is not necessarily attained. See Exercises 1.2.9 and 1.2.3. But if this is the case, any vector $x \in \mathcal{X}_1$, $x \neq 0$, for which

$$\|Ax\|_{\mathcal{X}_2} = \|A\|\,\|x\|_{\mathcal{X}_1}$$

is called a *maximizing vector* for A.

The operator $A \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ is called *lower bounded* or *bounded below* if there is a constant $c > 0$ such that

$$\|Ax\|_{\mathcal{X}_2} \geq c\|x\|_{\mathcal{X}_1} \qquad (x \in \mathcal{X}_1).$$

It is easy to see that A is lower bounded if A is left-invertible, that is, there is a bounded operator $B \in \mathcal{L}(\mathcal{X}_2, \mathcal{X}_1)$ such that $BA = I_{\mathcal{X}_1}$. The converse is true in the Hilbert space setting.

For a bounded operator $A : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$, the closed subspace

$$\ker A = \{x \in \mathcal{X}_1 : Ax = 0\}$$

is called the *kernel* of A. The kernel plays a major role in our discussion. Another set that frequently appears in this subject is the *range* of A,

$$\mathcal{R}(A) = \{Ax : x \in \mathcal{X}_1\},$$

which is a linear manifold of $\mathcal{X}_2$. Note that, contrary to the kernel, the range of a bounded operator is not necessarily closed. Nevertheless, if A is lower bounded, then the kernel of A is trivial (reduced to $\{0\}$) and the range of A is closed if $\mathcal{X}_1$ is a Banach space. Conversely, if A is one-to-one and has a closed range, and if $\mathcal{X}_1$ and $\mathcal{X}_2$ are Banach spaces, then A is lower bounded. This is a profound result and will be treated after studying the open mapping theorem (Corollary 1.17).

An operator $A \in \mathcal{L}(\mathcal{X})$ is said to be *power bounded* if there exists a constant $C > 0$ such that

$$\|A^n\| \le C \qquad (n \ge 0). \tag{1.4}$$

More restrictively, A is said to be *polynomially bounded* if there exists a constant $C > 0$ such that

$$\|p(A)\| \le C\|p\|_\infty \tag{1.5}$$

for every analytic polynomial p, where $\|p\|_\infty = \sup_{|z|=1} |p(z)|$. Clearly, a polynomially bounded operator is also power bounded. But the converse is not true.

A linear map from a normed linear space $\mathcal{X}$ into the complex plane $\mathbb{C}$ is called a *functional*. The family of all continuous functionals on $\mathcal{X}$ is called the *dual space* of $\mathcal{X}$ and is denoted by $\mathcal{X}^*$. The vector space $\mathcal{X}^*$ equipped with the operator norm

$$\|f\|_{\mathcal{X}^*} = \sup_{\|x\|_{\mathcal{X}} \le 1} |f(x)| \qquad (f \in \mathcal{X}^*)$$

is a Banach space. If there is no ambiguity, we will also write $\|f\|$ for $\|f\|_{\mathcal{X}^*}$. Characterizing the elements of $\mathcal{X}^*$ is an important theme in functional analysis, and it has profound applications. Since $\mathcal{X}^*$ is a normed linear space, we can equally consider the dual of $\mathcal{X}^*$, which is called the *second dual* of $\mathcal{X}$ and is naturally denoted by $\mathcal{X}^{**}$. The mapping

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}^{**} \\ x & \longmapsto & \hat{x}, \end{array}$$