1
The interplay of waves and turbulence: a preview

'Where shall we begin, please your majesty?' he asked. 'Begin at the beginning,' the King said, gravely, 'and go on till you come to the end: then stop.'

Lewis Carroll, Alice's Adventures in Wonderland

Large-scale geophysical and astrophysical flows are populated by a great variety of internal waves, some maintained by density stratification (internal gravity waves), some by the background planetary or stellar rotation (inertial waves), and yet others by the large-scale magnetic fields which thread through interplanetary space and are generated in the interiors of planets and stars (Alfvén waves). Sometimes these waves arise through a combination of factors, such as background rotation plus an ambient magnetic field (magnetostrophic waves), or a combination of rotation plus a sloping boundary (Rossby waves). Indeed, a bewildering variety of wave motions can be found in the oceans, the atmosphere, the core of the Earth and the Sun. These are rarely dynamically inert, but rather perform an important range of functions, not the least of which is their interaction with the turbulence which almost invariably accompanies such flows. Sometimes the waves excite new turbulence through instabilities; on other occasions they reshape the structure of the turbulence by dispersing its energy into particular patterns. Some turbulent flows seem to be strongly influenced by the action of waves (at least at large scales), while others seem largely impervious to the waves. So perhaps, before discussing turbulence, we should say something about these waves.

In this chapter we provide a qualitative discussion of waves, turbulence, and their mutual interaction. There is no attempt at mathematical rigour and we borrow freely from later chapters, anticipating many exact results. The aim is to provide a roadmap for the rest of the book.

1.1 Three types of wave

We shall focus on the three simplest types of internal wave found in geophysical and astrophysical flows:

(i) *inertial waves*, which are sustained by the Coriolis force associated with a background rotation;
(ii) *internal gravity waves*, which rely on a background gradient in density;
(iii) *Alfvén waves*, which propagate along magnetic field lines in an electrically conducting fluid.

We start with inertial waves, which rely on the fact that rotation can endow a fluid with a kind of internal elasticity; a most peculiar property which results in many bizarre and counterintuitive phenomena. Let us see if we can understand where this elasticity comes from. Suppose, for simplicity, that we have an incompressible, inviscid fluid of density ρ which rotates as a rigid body at the rate $\mathbf{\Omega} = \Omega\hat{\mathbf{e}}_z$. The fluid is perturbed and, in a frame of reference that rotates with $\mathbf{\Omega}$, this perturbation has velocity and pressure fields $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$. In this non-inertial frame of reference we must add to Newton's second law two fictitious forces; the Coriolis and centrifugal forces. These forces, evaluated per unit mass, can be written as $2\mathbf{u} \times \mathbf{\Omega}$ and $\nabla[\frac{1}{2}(\mathbf{\Omega} \times \mathbf{x})^2]$ respectively, and so the equation of motion (per unit mass) for the fluid in the rotating frame is:

$$\frac{D\mathbf{u}}{Dt} = -\nabla[p/\rho] + \nabla\left[\tfrac{1}{2}(\mathbf{\Omega} \times \mathbf{x})^2\right] + 2\mathbf{u} \times \mathbf{\Omega} \tag{1.1}$$

(acceleration = pressure force + centrifugal force + Coriolis force).

Here $-\nabla[p/\rho]$ is the net pressure force per unit mass acting on a fluid element, D/Dt is the usual convective derivative, and $D\mathbf{u}/Dt$ the acceleration of the fluid in the rotating frame. (The details of (1.1) are spelt out in §3.1.) Using p^* to denote the *reduced pressure*, $p - \frac{1}{2}\rho(\mathbf{\Omega} \times \mathbf{x})^2$, and recalling that $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ (see §2.1), this simplifies to

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla[p^*/\rho] + 2\mathbf{u} \times \mathbf{\Omega}. \tag{1.2}$$

From now on we shall drop the superscript * on p on the understanding that we are working with the reduced pressure.

Let us now restrict ourselves to small-amplitude perturbations about the state of rigid-body rotation. In particular, we consider the case where the dimensionless *Rossby number*,

$$\mathrm{Ro} = u/\Omega\ell, \tag{1.3}$$

is small, u and ℓ being typical scales of the motion in the rotating frame. Then $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is negligible by comparison with $2\mathbf{u} \times \mathbf{\Omega}$ and we can linearise the equation of motion to give

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla[p/\rho] + 2\mathbf{u} \times \mathbf{\Omega}, \tag{1.4}$$

whose curl is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = 2(\mathbf{\Omega} \cdot \nabla)\mathbf{u} = 2\Omega\frac{\partial \mathbf{u}}{\partial z}, \tag{1.5}$$

where $\boldsymbol{\omega}$ is the vorticity, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

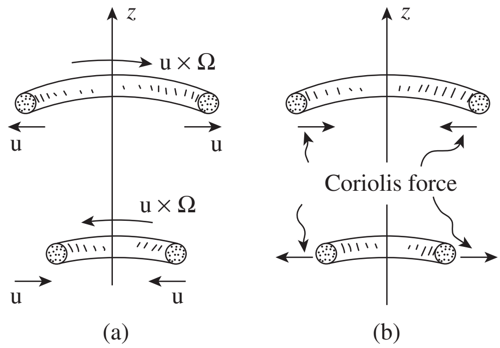

Figure 1.1 The effect of the Coriolis force on a loop of fluid. (a) A ring moving radially outward (top ring) induces a negative circulation about the loop, while one moving inward (bottom ring) induces a positive circulation. (b) The negative circulation in the top ring induces an inward Coriolis force which halts the expansion, while the positive circulation in the bottom ring induces an outward Coriolis force, countering the contraction.

Expressions (1.4) and (1.5) are the key equations for small-amplitude motion in a rapidly rotating fluid. When the motion is steady, (1.5) demands

$$\frac{\partial \mathbf{u}}{\partial z} = 0, \tag{1.6}$$

i.e. the flow is strictly two-dimensional in the sense that $\mathbf{u}$ is independent of z. This is known as the Taylor–Proudman theorem. Of particular importance is the observation that steady, low-Ro flows must satisfy $\partial u_z/\partial z = 0$, and since incompressibility demands $\nabla \cdot \mathbf{u} = 0$, this can be rewritten as

$$\nabla \cdot \mathbf{u}_\perp = -\partial u_z/\partial z = 0, \tag{1.7}$$

where $\mathbf{u}_\perp = (u_x, u_y, 0)$ is the motion in the lateral plane. Evidently, a strong background rotation tends to suppress the divergence of $\mathbf{u}$ in the horizontal plane, and in the case of steady motion this leads to $\partial \mathbf{u}/\partial z = 0$ and $\nabla \cdot \mathbf{u}_\perp = 0$.

So why does this lead to oscillations? Consider a circular hoop of fluid, c_m, sitting in the lateral, $x - y$, plane and executing axisymmetric motion, which we shall describe using cylindrical polar coordinates (r, θ, z). Suppose that the ring is initially stationary but is perturbed and starts to move radially outward ($u_r > 0$, $\nabla \cdot \mathbf{u}_\perp > 0$). Then the Coriolis force $2\mathbf{u} \times \boldsymbol{\Omega} = -2u_r \Omega \hat{\mathbf{e}}_\theta$ will induce a negative acceleration around the loop ($u_\theta < 0$), such that $\int \omega_z \mathrm{d}A = \oint_{c_\mathrm{m}} \mathbf{u} \cdot \mathrm{d}\mathbf{r} < 0$ (top ring in Figure 1.1(a)). The resulting negative value of ω_z is readily understood from Kelvin's theorem, provided we temporarily move back to an inertial frame of reference. In the inertial frame the axial vorticity is $2\Omega + \omega_z$ and Kelvin's theorem demands that the flux of vorticity through the material loop, c_m, is conserved (see Acheson, 1990, or else §2.5, for a discussion of Kelvin's theorem). Thus, if $A(t)$ is the area enclosed by c_m, we have $\int (2\Omega + \omega_z)\mathrm{d}A = 2\Omega A(t) + \int \omega_z \mathrm{d}A =$ constant. Evidently, as the loop expands, $\int \omega_z \mathrm{d}A$ must fall in order to conserve the flux in the inertial frame.

(A similar argument can be constructed in the inertial frame of reference using angular momentum conservation.) Returning now to the rotating coordinate system, this negative value of u_θ induces a second Coriolis force $2\mathbf{u} \times \mathbf{\Omega} = 2u_\theta \Omega \hat{\mathbf{e}}_r = -2|u_\theta|\Omega \hat{\mathbf{e}}_r$, which acts to reduce u_r and halt the expansion of the ring. This is shown in the top ring in Figure 1.1(b). Evidently the expansion must eventually cease and the ring will start to contract back to its initial position. However, inertia ensures an overshoot and the ring passes through its initial equilibrium position and continues to contract ($\nabla \cdot \mathbf{u}_\perp < 0$). The sequence of events is then repeated as described above, except that u_r is now negative and we find that the Coriolis force again acts to oppose the growth in $|\nabla \cdot \mathbf{u}_\perp|$, eventually halting the contraction of c_{m} and forcing the loop back towards its equilibrium position (Figure 1.1, bottom loop). And so it goes on, with the hoop oscillating back and forth.

In summary, then, strictly steady, low-Ro motion must satisfy $\partial \mathbf{u}/\partial z = 0$ and $\nabla \cdot \mathbf{u}_\perp = 0$. If there is some departure from $\nabla \cdot \mathbf{u}_\perp = 0$, the Coriolis force opposes this, and oscillations ensue (Batchelor, 1967). Let us see if we can quantify this behaviour. The vorticity equation (1.5), plus the divergence of the horizontal components of (1.4), yield

$$\frac{\partial \omega_z}{\partial t} = -2\Omega(\nabla \cdot \mathbf{u}_\perp), \tag{1.8}$$

and

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}_\perp) = 2\Omega\omega_z - \nabla_\perp^2(p/\rho), \tag{1.9}$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2 \partial y^2$. Evidently the horizontal expansion ($\nabla \cdot \mathbf{u}_\perp > 0$), or contraction ($\nabla \cdot \mathbf{u}_\perp < 0$), of a fluid element induces an axial component of vorticity in accordance with (1.8), with an expansion reducing ω_z and a contraction increasing ω_z. Turning now to (1.9), we see that a reduction (or growth) in ω_z tends to reduce (or increase) $\nabla \cdot \mathbf{u}_\perp$. Thus, as anticipated above, the Coriolis force provides a self-correcting mechanism, continually pushing any disturbance back towards the equilibrium condition $\nabla \cdot \mathbf{u}_\perp = 0$. Since inertia provides a mechanism for overshoot, this suggests the existence of waves oscillating about the equilibrium state $\nabla \cdot \mathbf{u}_\perp = 0$. Indeed, (1.8) and (1.9) yield

$$\frac{\partial^2}{\partial t^2}(\nabla \cdot \mathbf{u}_\perp) + (2\Omega)^2(\nabla \cdot \mathbf{u}_\perp) = \text{ pressure terms},$$

which hints at oscillations at an angular frequency of $\varpi = 2\Omega$. A more considered analysis, which incorporates the pressure forces, shows that plane waves of the form $\exp[\mathrm{j}(\mathbf{k} \cdot \mathbf{x} - \varpi t)]$ support a range of frequencies, ϖ, in which ϖ is related to the wavevector, $\mathbf{k}$, in accordance with the dispersion relationship

$$\varpi = \pm 2(\mathbf{k} \cdot \mathbf{\Omega})/|\mathbf{k}|. \tag{1.10}$$

(Note that, if we adopt the convention that $\varpi \geq 0$, then the positive sign in (1.10) corresponds to $k_z \geq 0$ and the negative sign to $k_z < 0$.) Evidently ϖ is independent of $|\mathbf{k}|$, but does depend on the angle between $\mathbf{k}$ and $\mathbf{\Omega}$. The derivation of (1.10) is provided in

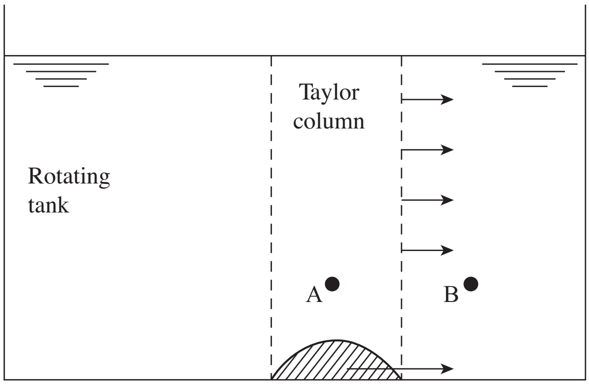

Figure 1.2 A small object of size R is towed slowly across the base of a rotating tank at a speed V. (From Davidson, 2004, by permission of Oxford University Press.)

§3.3.1, but in brief one applies the operator $(\partial/\partial t)\nabla\times$ to (1.5) to get the wave-like equation $(\partial^2/\partial t^2)(\nabla^2\mathbf{u}) + (2\Omega)^2\partial^2\mathbf{u}/\partial z^2 = 0$. The dispersion relationship then follows.

Clearly a range of frequencies is permissible, $0 \le \varpi \le 2\Omega$, with $\mathbf{k}$ perpendicular to $\boldsymbol{\Omega}$ yielding zero frequency, and $\mathbf{k}$ parallel to $\boldsymbol{\Omega}$ the highest frequency, $\varpi = 2\Omega$. In §3.3.1 it is shown that the corresponding group velocity (the velocity with which energy disperses in the form of wave packets) is given by

$$\mathbf{c}_g = \pm 2\mathbf{k} \times (\boldsymbol{\Omega} \times \mathbf{k})/|\mathbf{k}|^3. \tag{1.11}$$

Inertial waves have the remarkable property that the group velocity is perpendicular to $\mathbf{k}$, so that energy propagates parallel to the wave crests and *not* in the direction of travel of the crests. The low-frequency waves ($\mathbf{k}$ perpendicular to $\boldsymbol{\Omega}$) have a group velocity of $\mathbf{c}_g = \pm 2\boldsymbol{\Omega}/|\mathbf{k}|$, while the high-frequency waves ($\mathbf{k}$ parallel to $\boldsymbol{\Omega}$) have zero group velocity. Intermediate frequencies result in oblique energy propagation, with $\mathbf{c}_g$ non-parallel to $\boldsymbol{\Omega}$.

These inertial waves can give rise to some very odd effects, as can be seen from the following simple experiment, devised by G. I. Taylor and described in more detail in §3.2 and §3.3. Consider Figure 1.2, which shows an object being towed slowly across the base of a rapidly rotating tank. If V is the towing speed and R the characteristic size of the object, then we can quantify 'slow' as

$$\text{Ro} = \frac{V}{\Omega R} \ll 1.$$

Now suppose we treat the motion as quasi-steady. Then (1.5) demands $\partial u_z/\partial z = 0$, so that liquid columns cannot be stretched or compressed in the axial direction. It follows that the fluid cannot flow up and over the object as it drifts across the tank, as this would entail $\partial u_z/\partial z \neq 0$. The only possibility is that the cylindrical column of liquid located between the object and the free surface moves with the object, as if rigidly attached to it. The remaining fluid then flows around this imaginary cylinder in a two-dimensional pattern, consistent with $\partial u_z/\partial z = 0$. Thus the fluid at A moves across the tank, always centred above the object, while that at point B has to split into two streams in order to flow around the vertical cylinder that circumscribes the object. This apparently rigid column of fluid, which drifts across the tank attached to the object, is called a *Taylor column*; a truly remarkable phenomenon.

Of course, it is natural to ask how the fluid lying within the Taylor column knows to move with the object, keeping pace with it. The answer is inertial waves. As the object is towed across the tank the motion is not strictly steady, even though Ro is small. The object is then obliged to continually emit inertial waves, rather like a radio antenna. Since the motion of the object is slow, these are low-frequency waves, and we have already seen that such waves travel in the direction of $\mathbf{\Omega}$ and at a speed of $\mathbf{c}_g = 2\mathbf{\Omega}/|\mathbf{k}| \sim 2\Omega R$, where R is the size of the object. Since Ro is small, this speed is extremely fast by comparison with the towing rate, V. So the object continually emits waves which propagate upward and reach the free surface on a time-scale which is almost instantaneous by comparison with the towing time. It turns out that these waves carry the information that tells the fluid within the Taylor column to move horizontally, keeping pace with the object below. In short, our apparently rigid fluid column is continually formed and reformed by a train of inertial waves, and in many ways it *is* just an ensemble of inertial waves.

Let us now turn to internal gravity waves, which are maintained by a background gradient in density. Although more intuitively accessible, these too exhibit some unexpected, even bizarre, properties. Let us start by establishing the governing equations. As for inertial waves, we shall simplify matters by considering an incompressible, inviscid fluid. In the unperturbed state the fluid is stably stratified according to

$$\rho_0(z) = \bar{\rho} + \frac{\mathrm{d}\rho_0}{\mathrm{d}z} z, \qquad \frac{\mathrm{d}\rho_0}{\mathrm{d}z} < 0,$$

where $\bar{\rho}$ and $\mathrm{d}\rho_0/\mathrm{d}z$ are constants. The hydrostatic pressure distribution then satisfies

$$\nabla p_0 = \rho_0 \mathbf{g}, \qquad \mathbf{g} = -g\hat{\mathbf{e}}_z.$$

Now suppose that the equilibrium is perturbed, so that $\rho = \rho_0(z) + \rho'(\mathbf{x}, t)$. Incompressibility demands $\mathrm{D}\rho/\mathrm{D}t = 0$, and combined with mass conservation in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{u}] = 0,$$

we find that $\nabla \cdot \mathbf{u} = 0$ and

$$\frac{\mathrm{D}\rho'}{\mathrm{D}t} + \frac{\mathrm{d}\rho_0}{\mathrm{d}z} u_z = 0. \tag{1.12}$$

It is conventional to introduce the positive quantity N defined by

$$N^2 = -\frac{\mathrm{d}\rho_0}{\mathrm{d}z}\frac{g}{\bar{\rho}}, \tag{1.13}$$

which is a constant with the dimensions of s^{-1}. N is usually known as the *Väisälä–Brunt frequency.* In terms of N, the incompressibility condition becomes

$$\frac{D}{Dt}\left(\frac{g\rho'}{\bar{\rho}}\right) = N^2 u_z. \tag{1.14}$$

Turning now to dynamics, the equation of motion is

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho'\mathbf{g}, \tag{1.15}$$

where we have absorbed the hydrostatic pressure, p_0, into the definition of p, so that p is now understood to represent departures from the hydrostatic pressure distribution. We now invoke the *Boussinesq approximation*, which asserts that, if $|\rho - \bar{\rho}| \ll \bar{\rho}$, then we may replace the density on the left by the mean density, $\bar{\rho}$:

$$\bar{\rho}\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho'\mathbf{g}. \tag{1.16}$$

Let us now assume that the perturbations ρ' and $\mathbf{u}$ are small, so that quadratic terms in these small quantities may be neglected. Ignoring $\rho'|\mathbf{u}|$ in (1.14) and $|\mathbf{u}|^2$ in (1.16), D/Dt may be replaced by $\partial/\partial t$ and our governing equations simplify to

$$\frac{\partial}{\partial t}\left(\frac{g\rho'}{\bar{\rho}}\right) = N^2 u_z, \tag{1.17}$$

$$\bar{\rho}\frac{\partial\mathbf{u}}{\partial t} = -\nabla p + \rho'\mathbf{g}. \tag{1.18}$$

These are the governing equations for small-amplitude internal gravity waves.

Now it is clear that such a system may sustain waves, rather like surface gravity waves. If a fluid particle is displaced up or down, it will want to fall back to its equilibrium position. Some hint of this is obtained by combining the z-component of (1.18) with (1.17):

$$\frac{\partial^2 u_z}{\partial t^2} = -\frac{\partial}{\partial t}\left(\frac{g\rho'}{\bar{\rho}}\right) - \frac{\partial^2}{\partial t\partial z}\left(\frac{p}{\bar{\rho}}\right) = -N^2 u_z - \frac{\partial^2}{\partial t\partial z}\left(\frac{p}{\bar{\rho}}\right),$$

which we may rewrite as

$$\frac{\partial^2 u_z}{\partial t^2} + N^2 u_z = \text{pressure terms}. \tag{1.19}$$

This suggests oscillations at a frequency of $\varpi = N$. Moreover, in line with surface gravity waves, one might be tempted to suppose (incorrectly as it turns out) that such oscillations propagate horizontally. However, as with inertial waves, such an approach is too simplistic and the pressure terms need to be handled properly. An exact analysis of (1.17) and (1.18) is given in §4.4.1, where it is shown that a range of frequencies exist, governed by the dispersion relationship

$$\varpi = Nk_\perp/|\mathbf{k}|, \tag{1.20}$$

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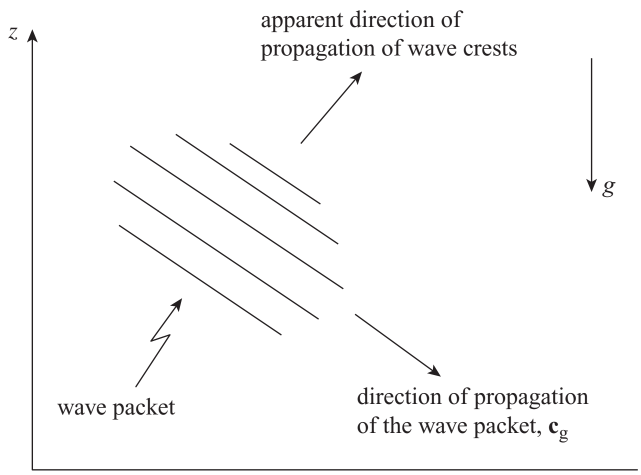

Figure 1.3 The energy of internal gravity waves propagates along the wave crests and not in the direction of travel of the crests. (From Davidson, 2004, by permission of Oxford University Press.)

where $k_\perp^2 = k_x^2 + k_y^2$. Evidently $0 < \varpi < N$ and, like inertial waves, ϖ is independent of $|\mathbf{k}|$, but does depend on the angle between $\mathbf{k}$ and $\hat{\mathbf{e}}_z$. The corresponding group velocity (the velocity with which energy disperses in the form of wave packets) is given by

$$\mathbf{c}_g = \frac{N}{|\mathbf{k}|^3 k_\perp}[\mathbf{k} \times (\mathbf{k} \times \mathbf{k}_{//})] = \frac{N}{|\mathbf{k}|^3 k_\perp}[k_{//}^2 \mathbf{k}_\perp - k_\perp^2 \mathbf{k}_{//}], \tag{1.21}$$

where $\mathbf{k}_{//} = \mathbf{k} - \mathbf{k}_\perp = (0, 0, k_z)$. (Again, see §4.4.1 for the details.) Evidently, wave energy need not propagate horizontally, but may travel in any direction. Moreover, just like inertial waves, internal gravity waves have the extraordinary property that energy propagates parallel to the wave crests and *not* in the direction of travel of the crests (Figure 1.3).

This time the low-frequency waves ($\varpi \approx 0$) have $\mathbf{k}$ aligned with $\hat{\mathbf{e}}_z$, while the high-frequency ones ($\varpi \approx N$) have $\mathbf{k}$ perpendicular to $\hat{\mathbf{e}}_z$. As with inertial waves, the low-frequency modes have the highest group velocity, only this time it is directed horizontally, while the high-frequency waves have zero group velocity. Thus vertically columnar disturbances ($k_z \ll k_\perp$) generate waves at a frequency of $\varpi \approx N$ which do not disperse, while flat, horizontal disturbances ($k_z \gg k_\perp$) initiate low-frequency waves ($\varpi \ll N$) which disperse energy efficiently in the horizontal plane and at a speed of $|\mathbf{c}_g| \sim N/|\mathbf{k}|$.

There is an analogue of Taylor columns in stratified flows, known as *blocking*. Consider the steady, two-dimensional flow of a stratified fluid past a cylindrical object, as shown in Figure 1.4. The oncoming flow has speed V, the cylinder radius R, and the flow lies in the x-z plane. We form the dimensionless *Froude number*,

$$\mathrm{Fr} = V/NR, \tag{1.22}$$

from V and R, and consider the situation where $\mathrm{Fr} \ll 1$; i.e. strongly stratified flow. In the limit of $\mathrm{Fr} \to 0$ the vertical velocity is suppressed, because there is very little kinetic

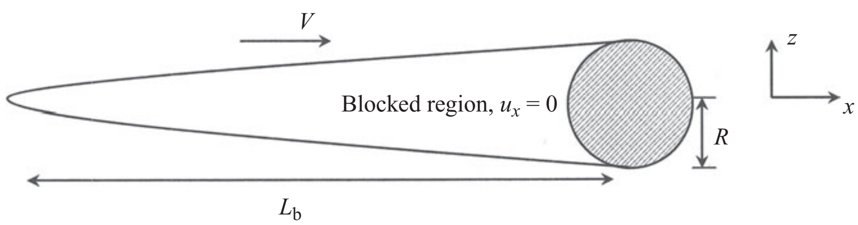

Figure 1.4 Stratified flow past a cylinder showing upstream blocking.

energy available to displace fluid particles from their equilibrium height (the potential energy barrier is too high). So the flow wants to follow almost horizontal paths. Moreover, this geometry demands that the flow is two-dimensional with $u_y = 0$, and so continuity requires $\partial u_x/\partial x \approx 0$. This is the stratified analogue of $\partial u_z/\partial z \approx 0$ in steady, rapidly-rotating flows. It follows that, for Fr $\to 0$, there should be a blocked region upstream and downstream of the cylinder, in which $u_x = 0$. These blocked regions are the analogue of the Taylor column shown in Figure 1.2.

In practice the flow pattern is more complex when Fr is small but finite, and when viscous effects are included (see Tritton, 1988). For finite values of Fr the downstream blocking tends to be lost, while a finite viscosity limits the upstream blocking to a region of length

$$L_b \sim \frac{\text{Re}}{\text{Fr}^2} R, \quad \text{Re} = \frac{VR}{\nu}, \tag{1.23}$$

as illustrated in Figure 1.4. (Here Re is the Reynolds number, ν the viscosity, and the derivation of (1.23) is given in §4.2.) Since the upstream blocked region is analogous to a Taylor column, and such columns are established by inertial wave propagation, we might anticipate that the blocked region shown in Figure 1.4 is the result of internal gravity waves. It turns out that this is correct, with the blocked region formed by low-frequency gravity waves which propagate upstream from the cylinder, confined to a horizontal plane.

So it seems that both inertial waves and internal gravity waves can give rise to extended, quasi-steady flow patterns, with rotation favouring axially elongated structures and stratification favouring flat, horizontal patterns. We shall see that something similar occurs within a cloud of turbulence, with columnar vortices dominating rapidly rotating turbulence, and flat, pancake-like vortices manifest in strongly stratified turbulence. However, the precise relationship between these turbulent, anisotropic vortices and internal wave propagation remains controversial.

Let us now turn to magnetohydrodynamics (MHD for short) and to our third category of wave: *Alfvén waves*. These propagate along magnetic field lines in electrically conducting fluids, such as astrophysical plasmas or the liquid core of many of the planets. As we shall see, in many ways Alfvén waves resemble the vibration of a plucked violin string, propagating as transverse waves at a speed determined by the tension in the field lines. It is, perhaps, inappropriate to delve into the theory of magnetohydrodynamics here (we do this in Chapters 5 and 6), so the following description is purely heuristic.

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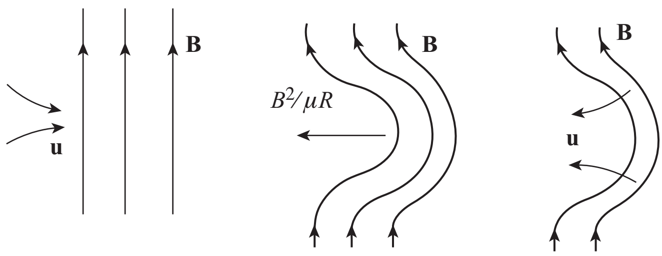

Figure 1.5 A perturbed magnetic field line exerts a force back on the fluid which tends to straighten the **B**-line.

Magnetic field lines in highly conducting fluids have two important properties. First, if they are bent out of shape they exert forces, called Lorentz forces, on the medium in which they sit. The easiest way to picture this is to imagine that magnetic field lines act like elastic bands, carrying a tension (called Faraday's tension) of magnitude $T = B^2/\mu$ per unit area. Here $B = |\mathbf{B}|$ is the magnetic field strength, $\mathbf{B}(\mathbf{x}, t)$ the magnetic field, and μ the permeability of free space. This tension gives rise to a force parallel to the field lines and, more importantly, to a force normal to the field lines, of magnitude $B^2/\mu R$ per unit volume, where R is the local radius of curvature of the **B**-lines. This is illustrated in Figure 1.5. The direction of this normal force is such as to reduce the curvature of the field-lines, and the more curved the field lines, the larger the force. So, if a magnetic field line is bent out of shape, it tries to snap back to a parallel configuration.

The second important property of magnetic fields is that, in highly-conducting fluids, they are 'frozen' into the medium, in the sense that fluid particles are glued to a particular field line and cannot part from it. So, if the fluid moves, it carries the magnetic field lines with it. This is a property which **B**-lines share with vortex lines in an inviscid, non-conducting fluid. Indeed, magnetic field lines in an infinitely conducting fluid, and vortex lines in a non-conducting inviscid fluid, are governed by identical evolution equations:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{u} \times \mathbf{B}], \quad \frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times [\mathbf{u} \times \boldsymbol{\omega}]. \tag{1.24}$$

Now consider the following situation. Suppose we have a uniform magnetic field with straight field lines sitting in a highly conducting fluid, say an astrophysical plasma. If there is a localised gust of fluid, say from left to right, then the **B**-lines will be bent out of shape by the flow, as illustrated in Figure 1.5. As the field lines bow out, they exert a progressively larger transverse force on the fluid, of magnitude $B^2/\mu R$. At some point this force is large enough to halt the left–right motion and reverse the flow, just like an elastic band. The fluid then flows backward, carrying the field lines back to their equilibrium position. However, it is inevitable that the field overshoots, since the field lines carry with them a certain mass of fluid, and so they have an effective inertia. The field then begins to bow out in the opposite direction, and the whole process starts in reverse. Oscillation is then inevitable.