Cambridge University Press
978-1-107-02685-8 - Canonical Ramsey Theory on Polish Spaces
Vladimir Kanovei, Marcin Sabok and Jindřich Zapletal
Excerpt
More information

1

Introduction

1.1 Motivation

The Ramsey theory starts with a classical result:

Fact 1.1 *For every partition of pairs of natural numbers into two classes there is a* homogeneous *infinite set: a set $a \subset \omega$ such that all pairs of natural numbers from a belong to the same class.*

It is not difficult to generalize this result for partitions into any finite number of classes. An attempt to generalize further, for partitions into infinitely many classes, hits an obvious snag: every pair of natural numbers could fall into its own class, and then certainly no infinite homogeneous set can exist for such a partition. Still, there seems to be a certain measure of regularity in partitions of pairs even into infinitely many classes. This is the beginning of canonical Ramsey theory.

Fact 1.2 (Erdős–Rado (Erdős and Rado 1950)) *For every equivalence relation E on pairs of natural numbers there is an infinite* homogeneous *set: a set $a \subset \omega$ on which one of the following happens:*

(i) $p \mathrel{E} q \leftrightarrow p = q$ *for all pairs* $p, q \in [a]^2$*;*
(ii) $p \mathrel{E} q \leftrightarrow \min(p) = \min(q)$ *for all pairs* $p, q \in [a]^2$*;*
(iii) $p \mathrel{E} q \leftrightarrow \max(p) = \max(q)$ *for all pairs* $p, q \in [a]^2$*;*
(iv) $p \mathrel{E} q$ *for all pairs* $p, q \in [a]^2$.

In other words, there are four equivalence relations on pairs of natural numbers such that any other equivalence can be *canonized*: made equal to one of the four equivalences on the set $[a]^2$, where $a \subset \omega$ is judiciously chosen infinite set. It is not difficult to see that the list of the four primal equivalence relations is irredundant: it cannot be shortened for the purposes of this theorem. It is also not difficult to see that the usual Ramsey theorems follow from the canonical version.

Further generalizations of these results can be sought in several directions. An exceptionally fruitful direction considers partitions and equivalences of substructures of a given finite or countable structure, such as in Nešetřil (2005). Another direction seeks to find homogeneous sets of larger cardinalities. In set theory with the axiom of choice, the search for uncountable homogeneous sets of arbitrary partitions leads to large cardinal axioms (Kanamori 1994), and this is one of the central concerns of modern set theory. A different approach will seek homogeneous sets for partitions that have a certain measure of regularity, typically expressed in terms of their descriptive set theoretic complexity in the context of Polish spaces (Todorcevic 2010). This is the path this book takes. Consider the following classical results:

Fact 1.3 (Mycielski 1964) *If R is a meager relation on a Polish space X, then there is a perfect set consisting of pairwise non-R-related elements.*

Fact 1.4 (Silver 1970) *For every partition $[\omega]^{\aleph_0} = B_0 \cup B_1$ into an analytic and coanalytic piece, one of the pieces contains a set of the form $[a]^{\aleph_0}$, where $a \subset \omega$ is some infinite set.*

Here, the space $[\omega]^{\aleph_0}$ of all infinite subsets of natural numbers is considered with the usual Polish topology which makes it homeomorphic to the space of irrational numbers. This is the most influential example of a Ramsey theorem on a Polish space. It deals with Borel partitions only as the Axiom of Choice can be easily used to construct a partition with no homogeneous set of the requested kind.

Are there any canonical Ramsey theorems on Polish spaces concerning sets on which Borel equivalence relations can be canonized? A classical example of such a theorem is the Silver dichotomy:

Fact 1.5 (Silver 1980) *If E is a coanalytic equivalence relation on a Polish space X, then either there is a perfect set consisting of pairwise E-related elements, or the space X decomposes into countably many E-equivalence classes.*

As a consequence, a coanalytic equivalence relation must have perfect set of pairwise inequivalent elements, or a perfect set of pairwise equivalent elements. If one wishes to obtain sets on which the equivalence relation is simple that are larger than just perfect, the situation becomes more complicated. Another classical result starts with an identification of Borel equivalence relations E_γ on the space $[\omega]^{\aleph_0}$ for every function $\gamma : [\omega]^{<\aleph_0} \to 2$ (the exact statement and definitions are stated in Section 8.3) and then proves the following:

Fact 1.6 (Prömel–Voigt, Mathias (Prömel and Voigt 1985; Mathias 1977)) *If $f : [\omega]^{\aleph_0} \to 2^\omega$ is a Borel function then there is γ and an infinite set $a \subset \omega$ such that for all infinite sets $b, c \subset a$, $f(b) = f(c) \leftrightarrow b \; E_\gamma \; c$.*

Thus, this theorem deals with *smooth* equivalence relations on the space $[\omega]^{\aleph_0}$, i. e., those equivalences E for which there is a Borel function $f : [\omega]^{\aleph_0} \to 2^\omega$ such that $b \; E \; c \leftrightarrow f(b) = f(c)$, and shows that such equivalence relations can be canonized to a prescribed form on a Ramsey cube. Other similar results can be found in the work of Otmar Spinas (Spinas 2001a,b; Klein and Spinas 2005). In the realm of nonsmooth equivalence relations, we have, for example:

Fact 1.7 (Connes–Feldman–Weiss (Connes *et al.* 1981)) *Suppose that X is a Polish space with an amenable countable Borel equivalence relation E on it and a quasi-invariant Borel probability measure μ. Then there is a Borel set $B \subset X$ of μ-mass 1 on which E is an orbit equivalence relation of a Borel action of $\mathbb{Z}$.*

The principal aim of this book is to expand this line of research in two directions. First, we consider canonization properties of equivalence relations more complicated than smooth in the sense of the Borel reducibility complexity rating of equivalence relations (Kanovei 2008; Gao 2009). It turns out that distinct complexity classes of equivalence relations possess various canonization properties, and most of the interest and difficulty lies in the nonsmooth cases. Second, we consider the task of canonizing the equivalence relations on Borel sets which are large from the point of view of various σ-ideals. Again, it turns out that σ-ideals commonly used in mathematical analysis greatly differ in their canonization properties, and a close relationship with their forcing properties, as described in Zapletal (2008), appears. Canonization theorems can then be applied to obtain ergodicity results for classical Borel equivalence relations such as E_2 and F_2.

1.2 Basic concepts

The central canonization notion of the book is the following:

Definition 1.8 A σ-ideal I on a Polish space X has *total canonization* for a class of equivalence relations if for every Borel I-positive set $B \subset X$ and every equivalence relation E on B in this class there is a Borel I-positive set $C \subset B$ consisting only of pairwise E-inequivalent elements or only of pairwise

E-equivalent elements. Total canonization (without the class of equivalence relations mentioned) means the total canonization for the class of all analytic equivalence relations.

The classes of equivalence relations considered in this book are nearly always closed under Borel reducibility; the broadest class would be that of all analytic equivalence relations. The total canonization is closely related to an ostensibly stronger notion:

Definition 1.9 A σ-ideal I on a Polish space X has the *Silver property* for a class of equivalence relations if for every Borel I-positive set $B \subset X$ and every equivalence relation E on B in this class, either there is a Borel I-positive set $C \subset B$ consisting of pairwise E-inequivalent elements, or B decomposes into a union of countably many E-equivalence classes and an I-small set. The Silver property (without the class of equivalence relations mentioned) means the Silver property for the class of all Borel equivalence relations.

Thus, the usual Silver theorem can be restated as the Silver property for coanalytic equivalence relations for the σ-ideal of countable sets. Unlike the total canonization, the Silver property introduces a true dichotomy, as the two options presented cannot coexist for σ-ideals containing all singletons: the Borel I-positive set C from the first option would have to have a positive intersection with one of the countably many equivalence classes from the second option, which is of course impossible. The Silver dichotomy also has consequences for undefinable sets. If there is *any* I-positive set $C \subset B$ consisting of pairwise E-inequivalent elements, then there must be a *Borel* such set, simply because the second option of the dichotomy is excluded by the same argument as above.

The Silver property certainly implies total canonization within the same class of equivalence relations: either there is a Borel I-positive set $B \subset C$ consisting of pairwise inequivalent elements, or one of the equivalence classes from the second option of the dichotomy must be I-positive and provides the second option of the total canonization. On the other hand, total canonization does not imply the Silver property as shown in Section 8.1. In many common cases, the various uniformization or game theoretic properties of the σ-ideal I in question can be used to crank up the total canonization to the Silver property. This procedure is described in Section 5.4 and it is the only way used to argue for the Silver property in this book.

There are a number of tricks used to obtain total canonization for various restricted classes of equivalence relations used in this book. However, the total canonization for all analytic equivalence relations is invariably proved via the following notion originating in Ramsey theory:

Definition 1.10 A σ-ideal I on a Polish space X has the *free set property* if for every I-positive analytic set $B \subset X$ and every analytic set $D \subset B \times B$ with all vertical sections in the ideal I there is a Borel I-positive set $C \subset B$ such that $(C \times C) \cap D \subseteq \text{id}$.

The free set property immediately implies the total canonization for analytic equivalence relations. If E is an analytic equivalence relation on an I-positive Borel set $B \subset X$, then either there is an I-positive E-equivalence class, which immediately yields the second option of total canonization, or, the relation $E \subset B \times B$ has I-small vertical sections, and the E-free I-positive set postulated by the free set property yields the first option of total canonization. It is not at all clear how one would argue for the opposite implication though, and we are coming to the first open question of this book.

Question 1.11 Is there a σ-ideal on a Polish space that has total canonization for analytic equivalence relations, but not the free set property?

The free set property is typically verified through fusion arguments as in Theorem 6.8, or through some version of the mutual generics property. This is the first place in this book where forcing makes explicit appearance through the following notion:

Definition 1.12 (Zapletal 2008) If I is a σ-ideal on a Polish space X, then the symbol P_I denotes the partial order of I-positive Borel subsets of X, ordered by inclusion.

Definition 1.13 A σ-ideal I on a Polish space X has the *mutual generics property* if for every Borel I-positive set B and every countable elementary submodel M of a large enough structure containing I and B there is a Borel I-positive subset $C \subset B$ such that its points are pairwise generic for the product forcing $P_I \times P_I$.

The mutual generics property is a strengthening of properness of the poset P_I, as the characterization of properness (Fact 2.50) shows. It implies the free set property by Proposition 2.57, but it is not implied by it by Theorem 8.1. It is somewhat ad hoc in that in many cases, the arguments demand that the product forcing is replaced by various reduced product forcings, see Sections 7.1 or 8.1. The resulting tools are very powerful and flexible, and where we cannot find them, we spend some effort proving that no version of mutual generics property can hold. The concept used to rule out all its versions is of independent interest:

Definition 1.14 A σ-ideal I on a Polish space X has a *square coding function* if there are a Borel I-positive set $B \subset X$ and a Borel function $f : B \times B \to 2^\omega$ such that for every analytic I-positive set $C \subset B$, we have $f''(C \times C) = 2^\omega$. The σ-ideal I has a *rectangular coding function* if there are Borel I-positive sets $B_0, B_1 \subset X$ and a Borel function $g : B_0 \times B_1 \to 2^\omega$ such that, for every analytic I-positive sets $C_0 \subset B_0$ and $C_1 \subset B_1$, we have $g''(C_0 \times C_1) = 2^\omega$.

Clearly, if a σ-ideal has a square coding function then no I-positive Borel set below the critical set B (which is always equal to the whole space in this book) can consist of points pairwise generic over a given countable model M, since some pairs in this set code via the function f, for example a transitive structure isomorphic to M. We will show that many σ-ideals have a square or rectangular coding function (Theorem 6.11 or 6.22) and that coding functions can be abstractly obtained from the failure of canonization of equivalence relations in many cases (Corollaries 7.35, 7.51, and 10.30).

In the cases where total canonization is unavailable, we can still provide a number of good canonization results. In the spirit of traditional Ramsey-theoretic notation, we introduce an arrow to record them. This is the central definition of the book:

Definition 1.15 Let **E**, **F** be two classes of equivalence relations, and let I be a σ-ideal on a Polish space X. $\mathbf{E} \to_I \mathbf{F}$ denotes the statement that for every I-positive Borel set $B \subset X$ and an equivalence $E \in \mathbf{E}$ on B there is a Borel I-positive set $C \subset B$ such that $E \restriction C \in \mathbf{F}$.

The strongest anti-canonization results we can achieve will be cast in terms of a *spectrum* of a σ-ideal. Note that it has no obvious counterpart in the canonical Ramsey theory on finite structures.

Definition 1.16 An analytic equivalence relation E is in the *spectrum* of a σ-ideal I on a Polish space X if there is an I-positive Borel set $B \subset X$ and an equivalence relation F on B which is Borel bireducible with E, and also for every I-positive Borel set $C \subset B$, $F \restriction C$ remains bireducible with E.

For example, the equivalence relations such as E_0 and F_2 are in the spectrum of the meager ideal, and E_{K_σ} is in the spectrum of the ideals associated with Silver forcing and Laver forcing. From the point of view of Ramsey theory, finding a nontrivial equivalence relation in the spectrum amounts to a strong negative result. However, results of this sort may be quite precious in themselves and have further applications.

1.3 Outline of results

For a good number of σ-ideals, we prove the strongest canonization results possible. As a motivational example, we include:

Theorem 1.17 (Corollary 6.16) *The σ-ideal on ω^ω σ-generated by compact sets has the Silver property. Restated, for every Borel equivalence relation E on ω^ω, exactly one of the following holds:*

(i) there is a closed set $C \subset \omega^\omega$ homeomorphic to ω^ω consisting of pairwise inequivalent points;
(ii) ω^ω is covered by countably many E-classes and countably many compact sets.

One satisfactory general theorem in this direction proved in this book states the following:

Theorem 1.18 (Theorem 6.8 and Corollary 6.9) *Let I be a σ-ideal on a compact space, σ-generated by a coanalytic family of compact sets. If I is calibrated, then it has the free set property, total canonization for analytic equivalence relations, and the Silver property.*

The class of calibrated σ-ideals introduced by Kechris and Louveau (1989) contains many σ-ideals commonly studied in abstract analysis: the σ-ideal on the unit interval σ-generated by closed sets of measure zero, the σ-ideal on the unit circle σ-generated by closed sets of uniqueness, or (up to a technical detail) the σ-ideal on the Hilbert cube σ-generated by compact sets of finite dimension.

In most cases, such a strong canonization result either is not available or we do not know how to prove it. Situations where the total canonization fails because there are clearly identifiable obstacles to it are of great interest. Therefore, we strive to canonize up to the known obstacles:

Theorem 1.19 (Theorem 9.3) *Let $n \in \omega$, let $\{I_i : i \in n\}$ be σ-ideals on respective compact spaces $\{X_i : i \in n\}$, σ-generated by coanalytic collection of compact sets and such that every I_i-positive analytic set has an I_i-positive compact subset. If $\{B_i : i \in n\}$ are Borel I_i-positive sets and E is an analytic equivalence relation on $\prod_i B_i$, there is a set $a \subset n$ and Borel I-positive sets $\{C_i \subset B_i : i \in n\}$ such that $E \restriction \prod_i C_i = \mathrm{id}_a$, where id_a is the equality on indices in the set a. In other words, writing I for the σ-ideal of those Borel subsets of $\prod_i X_i$ that do not contain a product of I_i-positive Borel sets, we have*

$$\text{analytic} \to_I \{\mathrm{id}_a : a \subset n\}.$$

Theorem 1.20 (Theorem 7.1) *Let I be the σ-ideal on 2^ω generated by Borel sets on which the equivalence relation E_0 is smooth. If B is a Borel I-positive set and E is an analytic equivalence relation, then there is a Borel I-positive set $C \subset B$ such that either C consists of pairwise inequivalent elements, or C consists of pairwise equivalent elements, or $E \restriction C = E_0$. In other words,*

$$\text{analytic} \rightarrow_I \{\text{id}, \text{ev}, E_0\}.$$

Theorem 1.21 (Corollary 7.7) *Let I be the σ-ideal on $(2^\omega)^\omega$ generated by Borel sets on which the equivalence relation E_1 is Borel reducible to E_0. If B is a Borel I-positive set and E is an a hypersmooth equivalence relation on B, then there is a Borel I-positive set $C \subset B$ such that either C consists of pairwise E-inequivalent elements, or C consists of pairwise E-equivalent elements, or $E \restriction C = E_1$. In other words,*

$$\text{hypersmooth} \rightarrow_I \{\text{id}, \text{ev}, E_1\}.$$

Theorem 1.22 (Theorem 7.36) *Let I be the σ-ideal on 2^ω generated by Borel sets on which the equivalence relation E_2 is essentially countable. If B is a Borel I-positive set and E is an an equivalence relation on B Borel reducible to $=_J$ for an F_σ P-ideal J on ω, then there is a Borel I-positive set $C \subset B$ such that either C consists of pairwise E-inequivalent elements, or C consists of pairwise E-equivalent elements, or $E \restriction C = E_2$. In particular,*

$$\ell_p \text{ equivalences} \rightarrow_I \{\text{id}, \text{ev}, E_2\}.$$

In a similar vein, we apply the canonization techniques to achieve a number of ergodicity results for certain classical Borel equivalence relations.

Definition 1.23 If E, F are equivalence relations on Polish spaces X and Y, we say that E is *F-generically ergodic* if for every Borel homomorphism from E to F there is a comeager subset of X that is mapped into a single F-equivalence class. Similarly, if μ is a Borel probability measure on X, we say that E is *μ, F-ergodic* if for every homomorphism from E to F, there is a subset of X of full μ-mass that is mapped into a single F-equivalence class.

Hjorth and Kechris (Kanovei 2008, theorem 13.5.3) showed that E_2 is F-generically ergodic for every equivalence relation F classifiable by countable structures. We replace their turbulence techniques with Ramsey theory and prove a similar result.

Theorem 1.24 (Theorem 6.66) *Let E be an equivalence relation on 2^ω; the space 2^ω is equipped with the usual Borel probability measure μ. If $E_2 \subseteq E$*

and E is classifiable by countable structures then E has a co-null class. Restated, E_2 is μ, F-ergodic for every equivalence relation F classifiable by countable structures.

Theorem 1.25 (Theorems 6.24 and 6.67) *Let E be an analytic equivalence relation on $(2^\omega)^\omega$; the space $(2^\omega)^\omega$ is equipped with the usual product topology and the product Borel probability measure μ. If $F_2 \subseteq E$ and F_2 is not Borel reducible to E, then E has a comeager class as well as a co-null class. Restated, for every analytic equivalence relation F exactly one of the following holds:*

(i) $F_2 \leq_B F$;
(ii) F_2 is F-generically ergodic and F_2 is μ, F-ergodic.

The weakest canonization results we obtain reduce the Borel reducibility complexity of equivalence relations to a specified class. It should be noted that no parallel to results of this kind exists in the realm of finite or countable canonization results.

Theorem 1.26 (Theorems 9.26 and 9.27) *Let E be an equivalence relation on $(2^\omega)^\omega$ which is classifiable by countable structures or Borel reducible to equality modulo an analytic P-ideal on ω. Then there are nonempty perfect sets $\langle P_n : n \in \omega\rangle$ such that $E \restriction \prod_n P_n$ is smooth. In other words, writing I for the σ-ideal of Borel subsets of $(2^\omega)^\omega$ which do not contain an infinite product of nonempty perfect sets,*

$$\text{classifiable by countable structures} \rightarrow_I \text{smooth},$$
$$(\leq_B =_J) \rightarrow_I \text{smooth}$$

whenever J is an analytic P-ideal on ω.

Theorem 1.27 (Theorem 8.17, originally by Mathias) *Let E be an essentially countable Borel equivalence relation on $[\omega]^{\aleph_0}$. Then there is an infinite set $a \subset \omega$ such that $E \restriction [a]^{\aleph_0}$ is Borel reducible to E_0.*

There is a number of anticanonization results. Again, the classification of analytic equivalence relations via their Borel reducibility complexity allows us to prove theorems that have no immediate counterpart in the finite or countable realm:

Theorem 1.28 (Corollary 6.55) *Whenever I is a σ-ideal such that P_I is proper and adds a dominating real, then E_{K_σ} is in the spectrum of I.*

Many results in the book either connect canonization properties of σ-ideals with the forcing properties of the associated quotient forcing P_I of Borel I-positive sets ordered by inclusion as studied in Zapletal (2008), or use that connection in their proofs. This allows us to tap a wealth of information amassed about such forcings by people otherwise not interested in descriptive set theory. Of special note is the connection between analytic equivalence relations and intermediate forcing extensions, explained in Chapter 3. It leads to the following general results:

Definition 1.29 Let E be an equivalence relation on a Polish space X, let I be a σ-ideal on X, and let $B \subset X$ be a set. We say that $E \restriction B$ is *I-ergodic* (or just *ergodic* if the σ-ideal is clear from the context) if for all Borel I-positive sets $C, D \subset B$ there are E-equivalent points $x \in C$, $y \in D$. The relation $E \restriction B$ is *nontrivially I-ergodic* any two Borel I-positive sets $C, D \subset B$ contain points $x \in C$ and $y \in D$ which are E-related and also points $x' \in C$ and $y' \in D$ that are E-unrelated.

Theorem 1.30 *Let I be a σ-ideal on a Polish space X such that the quotient poset P_I is proper.*

(i) (Theorem 3.1) *If P_I adds a minimal forcing extension, then for every I-positive Borel set $B \subset X$ and every analytic equivalence relation E on B there is a Borel I-positive $C \subset B$ such that $E \restriction C = \mathrm{id}$ or $E \restriction C$ is ergodic.*
(ii) (Corollary 4.10) *If P_I is nowhere c.c.c. (countable chain condition) and adds a minimal forcing extension then it has total canonization for equivalence relations classifiable by countable structures.*
(iii) (Theorem 4.9) *If P_I adds only finitely many real degrees then every equivalence relation classifiable by countable structures simplifies to an essentially countable equivalence relation on a Borel I-positive set.*

1.4 Navigation

Chapter 2 contains background material which at first glance has nothing to do with the canonization of equivalence relations. Most results are standard and stated without proofs, although there are exceptions such as the proof of canonical interpretation of $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ classes of analytic sets in generic extensions (Theorem 2.44). Section 2.6 describes the basic treatment of quotient posets as explored in Zapletal (2008). For a σ-ideal I on a Polish space X we write P_I for the poset of Borel I-positive subsets of X ordered by inclusion, and outline its basic forcing properties.