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978-1-107-02683-4 — Geometric and Topological Methods for Quantum Field Theory
Alexander Cardona, Iván Contreras, Andrés F. Reyes-Lega
Excerpt
More Information

Introduction

This volume offers an introduction to some recent developments in several active topics at the interface between geometry, topology, number theory and quantum field theory:

- new geometric structures, Poisson algebras and quantization,
- multizeta, polylogarithms and periods in quantum field theory,
- geometry of quantum fields and the standard model.

It is based on lectures and short communications delivered during a summer school on "Geometric and Topological Methods for Quantum Field Theory" held in Villa de Leyva, Colombia, in July 2009. This school was the sixth of a series of summer schools to take place in Colombia, which have taken place every other year since July 1999. The invited lectures, aimed at graduate students in physics or mathematics, start with introductory material before presenting more advanced results. Each lecture is self-contained and can be read independently of the others.

The volume begins with the introductory lectures on the geometry of Dirac structures by Henrique Bursztyn, in which the author provides the motivation, main features and examples of these new geometric structures in theoretical physics and their applications in Poisson geometry. These lectures are followed by an introduction to the geometry of holomorphic vector bundles over Riemann surfaces by Florent Schaffhauser, in which the author discusses the structure of spaces of connections, the notion of stability and takes us to the celebrated classification theorem of Donaldson for stable bundles. The third lecture, by Sylvie Paycha, explores possible extensions of the theory of characteristic classes and Chern–Weil theory to a class of infinite-dimensional bundles by means of pseudo-differential techniques. After some geometric preliminaries, the author presents the analytic tools (regularized traces and their properties) which are then used to extend the finite-dimensional Chern–Weil calculus to certain infinite-rank vector bundles, with a brief incursion in the Hamiltonian formalism in gauge theory.

The reader is led into the realm of perturbative quantum field theory with an introductory lecture by Stefan Weinzierl on the theory of Feynman integrals. Together with practical algorithms for evaluating Feynman integrals, the author discusses mathematical aspects of loop integrals related to periods, shuffle algebras and multiple polylogarithms. A further lecture by Francis Brown provides an introduction to recent work on iterated integrals and polylogarithms, with emphasis on the case of the thrice punctured Riemann sphere. The author also gives an overview of some recent results connecting such iterated integrals and polylogarithms with Feynman diagrams in perturbative quantum field theory.

In a subsequent lecture, Luis Boya discusses geometric structures that are relevant in quantum field theory and string theory. After an introduction to the basics of differential geometry aimed at physicists, the author discusses holonomy groups, higher-dimensional models relevant for string theory, M-theory and F-theory, as well as geometric aspects of compactification. The last lecture, by Florian Scheck, presents a critical account of some of the more puzzling aspects of the standard model, emphasizing phenomenological as well as geometric aspects. This includes a presentation of the basic geometric structures underlying gauge theories, a discussion of mass matrices and state mixing, a geometric account of anomalies and a review of the noncommutative geometry approach to the standard model. The lecture finishes with a discussion of spontaneous symmetry breaking based on causal gauge invariance.

The invited lectures are followed by four short communications on a wide spectrum of topics. In the first contribution, Leonardo Cano adapts some well-known techniques of spectral analysis for Schrödinger operators to the study of Laplacians on complete manifolds with corners of codimension 2. The author presents results on the absence of a singular continuous spectrum for such operators, as well as a description of the behavior of its pure point spectrum in terms of the underlying geometry. The chapter by Iván Contreras gives a categorical overview of the so-called formal groupoids and studies their associated Hopf algebroids, mentioning their relevance in the field of Poisson geometry as formal realizations of Poisson manifolds. Andrés Vargas presents in his contribution a detailed study of the Einstein condition on Riemannian manifolds with metrics of Hölder regularity, introducing the use of harmonic coordinates and considering the smoothness of the differentiable structure of the underlying manifold. Finally, in the last contribution, Alexander Cardona and César Del Corral study the index of Dirac-type operators associated to Atiyah–Patodi–Singer type boundary conditions from the point of view of weighted (super-)traces. The authors show that both the index of such an operator and the reduced eta-invariant term can be expressed in terms of weighted (super-) traces of identity operators determined by the boundary conditions.

We hope that these contributions will give – as much as the school itself seems to have given – young students the desire to pursue what might be their first acquaintance with some of the problems on the edge of mathematics and physics presented here. On the other hand, we hope that the more advanced reader will find some pleasure in reading about different outlooks on related topics and seeing how the well-known mathematical tools prove to be very useful in some areas of quantum field theory.

We are indebted to various organizations for their financial support for this school. Let us first of all thank Universidad de los Andes, which was our main source of financial support in Colombia. Other organizations such as CLAF (Centro Latino Americano de Fisica) in Brazil, and the International Mathematical Union, through the CDE (Commission on Development and Exchanges) program, also contributed in a substantial way to the financial support needed for this school.

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Without the people named here, all of whom helped in the organization in some way or another, before, during and after the school, this scientific event would not have left such vivid memories in the lecturers' and participants' minds. Last but not least, thanks to all the participants who gave us all, lecturers and editors, the impulse to prepare this volume through the enthusiasm they showed during the school, and thank you to all the contributors and referees for their participation in the realization of these proceedings.

The editors:

Alexander Cardona, Iván Contreras and Andrés Reyes-Lega

1

A brief introduction to Dirac manifolds

HENRIQUE BURSZTYN

Abstract

These lecture notes are based on a series of lectures given at the school on "Geometric and Topological Methods for Quantum Field Theory", in Villa de Leyva, Colombia. We present a basic introduction to Dirac manifolds, recalling the original context in which they were defined, their main features, and briefly mentioning more recent developments.

1.1 Introduction

Phase spaces of classical mechanical systems are commonly modeled by symplectic manifolds. It often happens that the dynamics governing the system's evolution are constrained to particular submanifolds of the phase space, e.g. level sets of conserved quantities (typically associated with symmetries of the system, such as momentum maps), or submanifolds resulting from constraints in the possible configurations of the system, etc. Any submanifold C of a symplectic manifold M inherits a *presymplectic* form (i.e. a closed 2-form, possibly degenerate), given by the pullback of the ambient symplectic form to C. It may be desirable to treat C in its own right, which makes presymplectic geometry the natural arena for the study of constrained systems; see e.g. [23, 25].

In many situations, however, phase spaces are modeled by more general objects: Poisson manifolds (see e.g. [35]). A Poisson structure on a manifold M is a bivector field $\pi \in \Gamma(\wedge^2 TM)$ such that the skew-symmetric bracket $\{f, g\} := \pi(df, dg)$ on $C^\infty(M)$ satisfies the Jacobi identity. Just as for symplectic phase spaces, there are natural examples of systems on Poisson phase spaces which are constrained

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to submanifolds. The present notes address the following motivating questions: what kind of geometric structure is inherited by a submanifold C of a Poisson manifold M? Can one "pullback" the ambient Poisson structure on M to C, in a similar way to what one does when M is symplectic? From another viewpoint, recall that M carries a (possibly singular) symplectic foliation, which completely characterizes its Poisson structure. Let us assume, for simplicity, that the intersection of C with each leaf $\mathcal{O}$ of M is a submanifold of C. Then $\mathcal{O} \cap C$ carries a presymplectic form, given by the pullback of the symplectic form on $\mathcal{O}$. So the Poisson structure on M induces a decomposition of C into presymplectic leaves. Just as Poisson structures define symplectic foliations, we can ask whether there is a more general geometric object underlying foliations with presymplectic leaves.

The questions posed in the previous paragraph naturally lead to *Dirac structures* [16, 17], a notion that encompasses presymplectic and Poisson structures. A key ingredient in the definition of a Dirac structure on a manifold M is the so-called *Courant bracket* [16] (see also [21]), a bilinear operation on the space of sections of $TM \oplus T^*M$ used to formulate a general integrability condition unifying the requirements that a 2-form is closed and that a bivector field is Poisson. These notes present the basics of Dirac structures, including their main geometric features and key examples. Most of the material presented here goes back to Courant's original paper [16], perhaps the only exception being the discussion about morphisms in the category of Dirac manifolds in Section 1.5.

Despite its original motivation in constrained mechanics,[1] recent developments in the theory of Dirac structures are related to a broad range of topics in mathematics and mathematical physics. Owing to space and time limitations, this chapter is *not* intended as a comprehensive survey of this fast growing subject (which justifies the omission of many worthy contributions from the references). A (biased) selection of recent aspects of Dirac structures is briefly sketched at the end of the chapter.

This chapter is structured as follows. In Section 1.2, we recall the main geometric properties of presymplectic and Poisson manifolds. Section 1.3 presents the definition of Dirac structures and their first examples. The main properties of Dirac structures are presented in Section 1.4. Section 1.5 discusses morphisms between Dirac manifolds. Section 1.6 explains how Dirac structures are inherited by submanifolds of Poisson manifolds. Section 1.7 briefly mentions some more recent developments and applications of Dirac structures.

[1] Dirac structures are named after Dirac's work on the theory of constraints in classical mechanics (see e.g. [20, 41]), which included a classification of constraint surfaces (first class, second class...), the celebrated Dirac bracket formula, as well as applications to quantization and field theory.

1.1.1 Notation, conventions, terminology

All manifolds, maps, vector bundles, etc. are smooth, i.e. in the C^∞ category. Given a smooth map $\varphi : M \to N$ and a vector bundle $A \to N$, we denote the pullback of A to M by $\varphi^* A \to M$.

For a vector bundle $E \to M$, a *distribution* D in E assigns to each $x \in M$ a vector subspace $D_x \subseteq E_x$. If the dimension of D_x, called the *rank* of D at x, is independent of x, we call the distribution *regular*. A distribution D in E is *smooth* if, for any $x \in M$ and $v_0 \in D_x$, there is a smooth local section v of E (defined on a neighborhood of x) such that $v(y) \in D_y$ and $v(x) = v_0$. A distribution that is smooth and regular is a subbundle. The rank of a smooth distribution is a lower semi-continuous function on M. For a vector bundle map $\Phi : E \to A$ covering the identity, the image $\Phi(E)$ is a smooth distribution of A; the kernel $\ker(\Phi)$ is a distribution of E whose rank is an upper semi-continuous function, so it is smooth if and only if it has locally constant rank. A smooth distribution D in TM is *integrable* if any $x \in M$ is contained in an *integral submanifold*, i.e. a connected immersed submanifold $\mathcal{O}$ so that $D|_{\mathcal{O}} = T\mathcal{O}$. An integrable distribution defines a decomposition of M into *leaves* (which are the maximal integral submanifolds); we generally refer to this decomposition of M as a *singular foliation*, or simply a *foliation*; see e.g. [22, Sec. 1.5] for details. When D is smooth and has constant rank, the classical Frobenius theorem asserts that D is integrable if and only if it is involutive. We refer to the resulting foliation in this case as *regular*.

Throughout the chapter, the Einstein summation convention is consistently used.

1.2 Presymplectic and Poisson structures

A symplectic structure on a manifold can be defined in two equivalent ways: either by a nondegenerate closed 2-form or by a nondegenerate Poisson bivector field. If one drops the nondegeneracy assumption, the first viewpoint leads to the notion of a *presymplectic* structure, while the second leads to *Poisson* structures. These two types of "degenerate" symplectic structures have distinct features that will be recalled in this section.

1.2.1 Two viewpoints on symplectic geometry

Let M be a smooth manifold. A 2-form $\omega \in \Omega^2(M)$ is called *symplectic* if it is nondegenerate and $d\omega = 0$. The nondegeneracy assumption means that the bundle map

$$\omega^\sharp : TM \to T^*M, \quad X \mapsto i_X\omega, \tag{1.1}$$

is an isomorphism; in local coordinates, writing $\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j$, this amounts to the pointwise invertibility of the matrix (ω_{ij}). The pair (M, ω), where ω is a symplectic 2-form, is called a *symplectic manifold*.

The basic ingredients of the Hamiltonian formalism on a symplectic manifold (M, ω) are as follows. For any function $f \in C^\infty(M)$, there is an associated *Hamiltonian vector field* $X_f \in \mathcal{X}(M)$, uniquely defined by the condition

$$i_{X_f}\omega = df. \tag{1.2}$$

In other words, $X_f = (\omega^\sharp)^{-1}(df)$. There is an induced bilinear operation

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

known as the *Poisson bracket*, that measures the rate of change of a function g along the Hamiltonian flow of a function f,

$$\{f, g\} := \omega(X_g, X_f) = \mathcal{L}_{X_f} g. \tag{1.3}$$

The Poisson bracket is skew-symmetric, and one verifies from its definition that

$$d\omega(X_f, X_g, X_h) = \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\}; \tag{1.4}$$

it follows that the Poisson bracket satisfies the Jacobi identity, since ω is closed. The pair $(C^\infty(M), \{\cdot,\cdot\})$ is a *Poisson algebra*, i.e. $\{\cdot,\cdot\}$ is a Lie bracket on $C^\infty(M)$ that is compatible with the associative commutative product on $C^\infty(M)$ via the Leibniz rule:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

It follows from the Leibniz rule that the Poisson bracket is defined by a bivector field $\pi \in \Gamma(\wedge^2 TM)$, uniquely determined by

$$\pi(df, dg) = \{f, g\} = \omega(X_g, X_f); \tag{1.5}$$

we write this locally as

$$\pi = \frac{1}{2}\pi^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}. \tag{1.6}$$

The bivector field π defines a bundle map

$$\pi^\sharp : T^*M \to TM, \quad \alpha \mapsto i_\alpha \pi, \tag{1.7}$$

in such a way that $X_f = \pi^\sharp(df)$. Since $df = \omega^\sharp(X_f) = \omega^\sharp(\pi^\sharp(df))$, we see that ω and π are related by

$$\omega^\sharp = (\pi^\sharp)^{-1} \quad \text{and} \quad (\omega_{ij}) = (\pi^{ij})^{-1}. \tag{1.8}$$

The whole discussion so far can be turned around, in that one can take the bivector field $\pi \in \Gamma(\wedge^2 TM)$, rather than the 2-form ω, as the starting point to define a symplectic structure. Given a bivector field $\pi \in \Gamma(\wedge^2 TM)$, we call it *nondegenerate* if the bundle map (1.7) is an isomorphism or, equivalently, if the local matrices (π^{ij}) in (1.6) are invertible at each point. We say that π is *Poisson* if the skew-symmetric bilinear bracket $\{f, g\} = \pi(df, dg)$, $f, g \in C^\infty(M)$, satisfies the Jacobi identity:

$$\mathrm{Jac}_\pi(f, g, h) := \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0, \tag{1.9}$$

for all $f, g, h \in C^\infty(M)$.

The relation

$$\pi(df, dg) = \omega(X_g, X_f)$$

establishes a 1–1 correspondence between nondegenerate bivector fields and non-degenerate 2-forms on M, in such a way that the bivector field is Poisson if and only if the corresponding 2-form is closed (see (1.4)). So a symplectic manifold can be equivalently defined as a manifold M equipped with a nondegenerate bivector field π that is Poisson.

The two alternative viewpoints to symplectic structures are summarized in the following table:

Nondegenerate $\pi \in \Gamma(\wedge^2 TM)$	Nondegenerate $\omega \in \Omega^2(M)$
$\mathrm{Jac}_\pi = 0$	$d\omega = 0$
$X_f = \pi^\sharp(df)$	$i_{X_f}\omega = df$
$\{f, g\} = \pi(df, dg)$	$\{f, g\} = \omega(X_g, X_f)$

Although the viewpoints are interchangeable, one may turn out to be more convenient than the other in specific situations, as illustrated next.

1.2.2 Going degenerate

There are natural geometric constructions in symplectic geometry that may spoil the nondegeneracy condition of the symplectic structure, and hence take us out of the symplectic world. We mention two examples.

Consider the problem of passing from a symplectic manifold M to a submanifold $\iota : C \hookrightarrow M$. To describe the geometry that C inherits from M, it is more natural to represent the symplectic structure on M by a 2-form ω, which can then be pulled back to C. The resulting 2-form $\iota^*\omega$ on C is always closed, but generally fails to be nondegenerate.

As a second example, suppose that a Lie group G acts on a symplectic manifold M by symmetries, i.e. preserving the symplectic structure, and consider the geometry inherited by the quotient M/G (we assume, for simplicity, that the action is free and proper, so the orbit space M/G is a smooth manifold). In this case, it is more convenient to think of the symplectic structure on M as a Poisson bivector field π, which can then be projected, or pushed forward, to M/G since π is assumed to be G-invariant. The resulting bivector field on M/G always satisfies (1.9), but generally fails to be nondegenerate.

These two situations illustrate why one may be led to generalize the notion of a symplectic structure by dropping the nondegeneracy condition, and how there are two natural ways to do it. Each way leads to a different kind of geometry: a manifold equipped with a closed 2-form, possibly degenerate, is referred to as *presymplectic*, while a *Poisson manifold* is a manifold equipped with a Poisson bivector field, not necessarily nondegenerate. The main features of presymplectic and Poisson manifolds are summarized below.

Presymplectic manifolds

On a presymplectic manifold (M, ω), there is a natural *null distribution* $K \subseteq TM$, defined at each point $x \in M$ by the kernel of ω:

$$K_x := \ker(\omega)_x = \{X \in T_xM \mid \omega(X, Y) = 0 \; \forall \; Y \in T_xM\}.$$

This distribution is not necessarily regular or smooth. In fact, K is a smooth distribution if and only if it has locally constant rank (see Section 1.1.1). For $X, Y \in \Gamma(K)$, note that

$$i_{[X,Y]}\omega = \mathcal{L}_X i_Y \omega - i_Y \mathcal{L}_X \omega = \mathcal{L}_X i_Y \omega - i_Y(i_X d + d i_X)\omega = 0;$$

it follows that, when K is regular, it is integrable by Frobenius' theorem. We refer to the resulting regular foliation tangent to K as the *null foliation* of M.

One may still define Hamiltonian vector fields on (M, ω) via (1.2), but, without the nondegeneracy assumption on ω, there might be functions admitting no Hamiltonian vector fields (e.g. if df lies outside the image of (1.1) at some point). We say that a function $f \in C^\infty(M)$ is *admissible* if there exists a vector field X_f such that (1.2) holds. In this case, X_f is generally not uniquely defined, as we may change it by the addition of any vector field tangent to K. Still, the Poisson bracket formula

$$\{f, g\} = \mathcal{L}_{X_f} g \tag{1.10}$$

is well defined (i.e. independent of the choice of X_f) when f and g are admissible. Hence the space of admissible functions, denoted by

$$C^\infty_{\mathrm{adm}}(M) \subseteq C^\infty(M),$$

is a Poisson algebra.

When K is regular, a function is admissible if and only if $df(K) = 0$, i.e. f is constant along the leaves of the null foliation; in particular, depending on how complicated this foliation is, there may be very few admissible functions (e.g. if there is a dense leaf, only the constant functions are admissible). When K is regular and the associated null foliation is simple, i.e. the leaf space M/K is smooth and the quotient map $q : M \to M/K$ is a submersion, then M/K inherits a symplectic form ω_{red}, uniquely characterized by the property that $q^*\omega_{red} = \omega$; in this case, the Poisson algebra of admissible functions on M is naturally identified with the Poisson algebra of the symplectic manifold $(M/K, \omega_{\mathrm{red}})$ via

$$q^* : C^\infty(M/K) \stackrel{\sim}{\to} C^\infty_{\mathrm{adm}}(M)$$

(see e.g. [37, Sec. 6.1] and references therein).

Poisson manifolds

If (M, π) is a Poisson manifold, then any function $f \in C^\infty(M)$ defines a (unique) Hamiltonian vector field $X_f = \pi^\sharp(df)$, and the whole algebra of smooth functions $C^\infty(M)$ is a Poisson algebra with bracket $\{f, g\} = \pi(df, dg)$.

The image of the bundle map $\pi^\sharp$ in (1.7) defines a distribution on M,

$$R := \pi^\sharp(T^*M) \subseteq TM, \qquad (1.11)$$

not necessarily regular, but always smooth and integrable. (The integrability of the distribution R may be seen as a consequence of Weinstein's splitting theorem [43].) So it determines a singular foliation of M, in such a way that two points in M lie in the same leaf if and only if one is accessible from the other through a composition of local Hamiltonian flows. One may verify that the bivector field π is "tangent to the leaves", in the sense that, if $f \in C^\infty(M)$ satisfies $\iota^* f \equiv 0$ for a leaf $\iota : \mathcal{O} \hookrightarrow M$, then $X_f \circ \iota \equiv 0$. So there is an induced Poisson bracket $\{\cdot, \cdot\}_{\mathcal{O}}$ on $\mathcal{O}$ determined by

$$\{f \circ \iota, g \circ \iota\}_{\mathcal{O}} := \{f, g\} \circ \iota, \qquad f, g \in C^\infty(M),$$

which is nondegenerate; in particular, each leaf carries a symplectic form, and one refers to this foliation as the *symplectic foliation* of π. The symplectic foliation of a Poisson manifold uniquely characterizes the Poisson structure. For more details and examples, see e.g. [11, 22, 35].

Remark 1 The integrability of the distribution (1.11) may be also seen as resulting from the existence of a Lie algebroid structure on T^*M, with anchor $\pi^\sharp : T^*M \to TM$ and Lie bracket on $\Gamma(T^*M) = \Omega^1(M)$ uniquely characterized by

$$[df, dg] = d\{f, g\},$$

see e.g. [11, 15]; we will return to Lie algebroids in Section 1.4.