# Lectures on Real Analysis

This is a rigorous introduction to real analysis for undergraduate students, starting from the axioms for a complete ordered field and a little set theory. The book avoids any preconceptions about the real numbers and takes them to be nothing but the elements of a complete ordered field. All of the standard topics are included, as well as a proper treatment of the trigonometric functions, which many authors take for granted. The final chapters of the book provide a gentle, example-based introduction to metric spaces with an application to differential equations on the real line.

The author's exposition is concise and to the point, helping students focus on the essentials. Over 200 exercises of varying difficulty are included, many of them adding to the theory in the text. The book is ideal for second-year undergraduates and for more advanced students who need a foundation in real analysis.

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# Contents

# Preface

This book is a rigorous introduction to real analysis, suitable for a one-semester course at the second-year undergraduate level, based on my experience of teaching this material many times in Australia and Canada. My aim is to give a treatment that is brisk and concise, but also reasonably complete and as rigorous as is practicable, starting from the axioms for a complete ordered field and a little set theory.

Along with epsilons and deltas, I emphasise the alternative language of neighbourhoods, which is geometric and intuitive and provides an introduction to topological ideas. I have included a proper treatment of the trigonometric functions. They are sophisticated objects, not to be taken for granted. This topic is an instructive application of the theory of power series and other earlier parts of the book. Also, it involves the concept of a group, which most students won't have seen in the context of analysis before.

There may be some novelty in the gentle, example-based introduction to metric spaces at the end of the book, emphasising how straightforward the generalisation of many fundamental notions from the real line to metric spaces really is. The goal is to develop just enough metric space theory to be able to prove Picard's theorem, showing how a detour through some abstract territory can contribute back to analysis on the real line.

Needless to say, I claim no originality whatsoever for the material in this book. My contribution, such as it is, lies in the selection and presentation of the material. I thank the American Mathematical Society for allowing the book to be formatted with one of their class files.

Finnur Lárusson

# To the student

The purpose of this course is twofold. First, to give a careful treatment of calculus from first principles. In first-year calculus we learn methods for solving specific problems. We focus on how to use these methods more than why they work. To pave the way for further studies in pure and applied mathematics we need to deepen our understanding of *why*, as opposed to *how*, calculus works. This won't be a simple rehashing of first-year calculus at all. Calculus done this way is called *real analysis*.

In particular, we will consider what it is about the real numbers that makes calculus work. Why can't we make do with the rationals? We will identify the key property of the real numbers, called *completeness*, that distinguishes them from the rationals and permeates all of mathematical analysis. Completeness will be our main theme through the whole course.

The second goal of the course is to practise reading and writing mathematical proofs. The course is proof-oriented throughout, not to encourage pedantry, but because proof is the only way that mathematical truth can be known with certainty. Mathematical knowledge is accumulated through long chains of reasoning. We can't rely on this knowledge unless we're sure that every link in the chain is sound. In many future endeavours, you will find that being able to construct and communicate solid arguments is a very useful skill.

With the emphasis on rigorous arguments comes the need to make our fundamental assumptions, from which our reasoning begins, clear and explicit. We shall list ten axioms that describe the real numbers and that can in fact be shown to *characterise* the real numbers. Our development of real analysis will be based on these axioms, along with a bit of set theory.

Towards the end of the course we extend some of the concepts we will have developed in the context of the real numbers to the much more general setting of metric spaces. To demonstrate the power of abstraction, the course ends with the proof, using metric space theory, of an existence and uniqueness theorem for solutions of differential equations.