CAMBRIDGE TRACTS IN MATHEMATICS

General Editors

B. BOLLOBÁS, W. FULTON, F. KIRWAN,
P. SARNAK, B. SIMON, B. TOTARO

209 Non-homogeneous Random Walks

CAMBRIDGE TRACTS IN MATHEMATICS

A complete list of books in the series can be found at www.cambridge.org/mathematics.
Recent titles include the following:

174. Forcing Idealized. By J. ZAPLETAL
175. The Large Sieve and its Applications. By E. KOWALSKI
176. The Monster Group and Majorana Involutions. By A. A. IVANOV
177. A Higher-Dimensional Sieve Method. By H. G. DIAMOND, H. HALBERSTAM, and W. F. GALWAY
178. Analysis in Positive Characteristic. By A. N. KOCHUBEI
179. Dynamics of Linear Operators. By F. BAYART and É. MATHERON
180. Synthetic Geometry of Manifolds. By A. KOCK
181. Totally Positive Matrices. By A. PINKUS
182. Nonlinear Markov Processes and Kinetic Equations. By V. N. KOLOKOLTSOV
183. Period Domains over Finite and *p*-adic Fields. By J.-F. DAT, S. ORLIK, and M. RAPOPORT
184. Algebraic Theories. By J. ADÁMEK, J. ROSICKÝ, and E. M. VITALE
185. Rigidity in Higher Rank Abelian Group Actions I: Introduction and Cocycle Problem. By A. KATOK and V. NIŢICĂ
186. Dimensions, Embeddings, and Attractors. By J. C. ROBINSON
187. Convexity: An Analytic Viewpoint. By B. SIMON
188. Modern Approaches to the Invariant Subspace Problem. By I. CHALENDAR and J. R. PARTINGTON
189. Nonlinear Perron-Frobenius Theory. By B. LEMMENS and R. NUSSBAUM
190. Jordan Structures in Geometry and Analysis. By C.-H. CHU
191. Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion. By H. OSSWALD
192. Normal Approximations with Malliavin Calculus. By I. NOURDIN and G. PECCATI
193. Distribution Modulo One and Diophantine Approximation. By Y. BUGEAUD
194. Mathematics of Two-Dimensional Turbulence. By S. KUKSIN and A. SHIRIKYAN
195. A Universal Construction for Groups Acting Freely on Real Trees. By I. CHISWELL and T. MÜLLER
196. The Theory of Hardy's Z-Function. By A. IVIĆ
197. Induced Representations of Locally Compact Groups. By E. KANIUTH and K. F. TAYLOR
198. Topics in Critical Point Theory. By K. PERERA and M. SCHECHTER
199. Combinatorics of Minuscule Representations. By R. M. GREEN
200. Singularities of the Minimal Model Program. By J. KOLLÁR
201. Coherence in Three-Dimensional Category Theory. By N. GURSKI
202. Canonical Ramsey Theory on Polish Spaces. By V. KANOVEI, M. SABOK, and J. ZAPLETAL
203. A Primer on the Dirichlet Space. By O. EL-FALLAH, K. KELLAY, J. MASHREGHI, and T. RANSFORD
204. Group Cohomology and Algebraic Cycles. By B. TOTARO
205. Ridge Functions. By A. PINKUS
206. Probability on Real Lie Algebras. By U. FRANZ and N. PRIVAULT
207. Auxiliary Polynomials in Number Theory. By D. MASSER
208. Representations of Elementary Abelian *p*-Groups and Vector Bundles. By D. J. BENSON
209. Non-homogeneous Random Walks. By M. MENSHIKOV, S. POPOV, and A. WADE

Non-homogeneous Random Walks

Lyapunov Function Methods for Near-Critical Stochastic Systems

MIKHAIL MENSHIKOV
University of Durham

SERGUEI POPOV
Universidade Estadual de Campinas, Brazil

ANDREW WADE
University of Durham

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107026698
10.1017/9781139208468

First published 2017

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Menshikov, M. V. (Mikhail Vasilevich) | Popov, Serguei, 1972– |
Wade, Andrew (Andrew R.), 1981–
Title: Non-homogeneous random walks : Lyapunov function methods for
near-critical stochastic systems / Mikhail Menshikov, University of Durham,
Serguei Popov, Universidade Estadual de Campinas,
Brazil, Andrew Wade, University of Durham.
Description: Cambridge : Cambridge University Press, 2017. |
Includes bibliographical references and index.
Identifiers: LCCN 2016036262 | ISBN 9781107026698 (hardback : alk. paper)
Subjects: LCSH: Random walks (Mathematics) | Stochastic processes.
Classification: LCC QA274.73.M46 2017 | DDC 519.2/82–dc23
LC record available at https://lccn.loc.gov/2016036262

ISBN 978-1-107-02669-8 Hardback

Contents

Preface

What Is This Book About?

This book has two main goals:

- to give an up-to-date exposition of the 'semimartingale' or 'Lyapunov function' approach to the analysis of stochastic processes;
- to present applications of the methodology to fundamental models (classical and modern) in probability theory and related fields.

Our expository bridge between these dual aims, between *methods* and *models*, is the d-dimensional *non-homogeneous random walk*, which as a model is simple to describe, closely resembling the classical homogeneous random walk, but which displays many interesting and subtle phenomena alien to the classical model. Non-homogeneous random walks cannot be studied by the techniques generally used for homogeneous random walks: new methods (and, just as importantly, new intuitions) are required.

Semimartingale and Lyapunov function ideas lead to a unified and powerful methodology in this context. As well as non-homogeneous random walks, we present applications of the methods to several other models from modern probability theory; while any of the models that we discuss can be studied by several probabilistic techniques, we believe that only the Lyapunov function method has something to say about *all* of them.

We emphasize that semimartingale methods are 'robust' in the sense that the underlying stochastic process need not satisfy simplifying assumptions such as the Markov property, reversibility, or time homogeneity, for instance, and the state space of the process need not be countable. In such a general setting, the semimartingale approach has few rivals. In particular, the methods presented work for *non-reversible* Markov chains. A general feeling is that, if a Markov chain is reversible, then things can be done in many possible ways:

there are methods from electrical networks, spectral calculations, harmonic analysis, etc. On the other hand, the non-reversible case is usually much harder. Similarly, the Markovian setting is not essential to the methods. In the semimartingale approach, the Markov property is a side issue and non-Markovian processes can be treated equally well.

The Lyapunov function approach for analysis of Markov processes originated with classical work of Foster, and the theory has since expanded greatly and proved very successful in analysis of numerous Markov models. Aspects of Foster–Lyapunov theory are presented in [6, 96, 239] (the presentation in [96] being the closest to our perspective). However, almost all of these existing presentations are concerned with the situation in which the process under consideration is not too close to a phase boundary in terms of its asymptotic behaviour. This book deals with analysis of *near-critical* systems, which exhibit fundamental phase transitions such as that between recurrence and transience.

Near-critical systems are exactly that: even if transient, they are not ballistic; even if positive recurrent, they do not exhibit geometric ergodicity; random quantities associated with the system typically have heavy (power-law) tails. Heavy tails have become increasingly prevalent in applications across many fields over the last few years, including queueing theory and finance; physicists associate heavy tails with the presence of ‘self-organized criticality’, a very fashionable topic at the moment. Naturally, the analysis of near-critical systems is more challenging and delicate than that for systems that are far from criticality.

The fundamental contributions to the near-critical situation come from classical work of Lamperti, and almost none of this has previously appeared in any book, despite its age and importance. Lamperti’s basic problem concerned the asymptotic analysis of a stochastic process on the half-line with mean drift at x of order $1/x$: this is exactly the critical situation in respect of the recurrence classification of the process. Importantly, Lamperti’s techniques are based on semimartingale ideas, and so the Markov property is not essential. Building on Lamperti’s ideas, over the last 15 years much work has appeared in academic journals on semimartingale methods and near-critical probabilistic systems, and this is the theory that we present in this book.

We want to emphasize the importance of applications of the theory. The Lyapunov function ideology often enables one to reduce a question about a complex model arising in applications to a question about a simpler one-dimensional model by considering a function of the original process whose image is one-dimensional. If the original model is interesting, in that its behaviour is near-critical in some sense, then the image process (for a suitably chosen Lyapunov function) will be near-critical. To give a concrete example, the classical and fundamental model of symmetric simple random walk on $\mathbb{Z}^d$ (X_n, say) can be analysed through the Lyapunov function $f(\mathbf{x}) = \|\mathbf{x}\|$ (the Euclidean norm). Then $f(X_n)$ is a stochastic process on the half-line with mean drift of order $1/x$ at x, i.e., precisely a critical Lamperti-type process. Moreover, $f(X_n)$ is *not* a Markov process, demonstrating the importance of the generality of the semimartingale approach.

Much more complicated and non-classical examples can be studied by the same methods, and we present several such examples to give a flavour of the power and utility of the techniques. We mention, for example, models from queueing theory, interacting particle systems, random walks in random environments, and so on. As mentioned above, our canonical example will be the non-homogeneous random walk. This model is a natural generalization of the very classical and extremely well-studied homogeneous random walk, but whose analysis requires entirely new methods. The semimartingale approach gives a systematic and intuitive way to analyse these processes.

So, to summarize, this book is about the analysis of Markov processes (such as random walks) via the method of Lyapunov functions; a correctly applied Lyapunov function of a process gives rise to a semimartingale. Our terminology here is neither particularly standard nor particularly precise; we discuss briefly here our usage.

What Is a Random Walk?

Many random walks are sums of i.i.d. random variables (or vectors); this usage is too narrow for us. Many random walks take place on graphs or groups; combinatorial or algebraic considerations are not the focus of this book. Our random walks are (usually) Markov chains, on state spaces that are embedded in Euclidean space, with transitions that are in some sense local (so that it is natural to speak of ‘jumps’ or ‘increments’). These are the models that are best suited to the probabilistic approach of this book, and include broad classes of models of interest in applications, such as queueing theory or ecology.

This book is not just about random walks; we discuss other Markov processes, including interacting particle systems and a stochastic billiards model, for example, but random walks provide a rich set of models on which to demonstrate some aspects of the Lyapunov function method.

What Is a Lyapunov Function?

The phrase 'Lyapunov function' has a quite precise technical meaning in the theory of stability of differential equations. For us, it has a much looser meaning: a Lyapunov function earns the name if the image under the function of a stochastic process is a process satisfying some conditions that enable one to deduce some property of the original process. For instance, if $(\xi_n, n \geq 0)$ is a time-homogeneous Markov chain on $\mathbb{Z}^d$, and $f : \mathbb{R}^d \to \mathbb{R}$ is a judicious choice of function such that there is a set $A \subset \mathbb{Z}^d$ for which

$$\mathbb{E}[f(\xi_{n+1}) - f(\xi_n) \mid \xi_n = \mathbf{x}] \leq 0, \text{ for all } \mathbf{x} \notin A, \qquad (\star)$$

then one can deduce that ξ_n is recurrent, or transient, if f and A satisfy certain simple conditions. Results of this kind, which have their origins in work of F. G. Foster in the 1950s, are known as Foster–Lyapunov conditions.

In all but the simplest cases, one cannot compute the left-hand side of $(\star)$ exactly, but often one can estimate $(\star)$ using Taylor's formula and coarse properties of the increments $\xi_{n+1} - \xi_n$; usually one or two moments suffice. Truncation arguments may be needed to control unusually large increments, where Taylor's formula will break down.

The language and tools of stopping times and martingale theory are close at hand. If $\tau = \min\{n \geq 0 : \xi_n \in A\}$ denotes the first hitting time of A, the *drift condition* $(\star)$ above can be interpreted as saying that $f(\xi_{n \wedge \tau})$ is a *supermartingale* adapted to the natural filtration. Since recurrence and transience properties of ξ_n can be related to properties of the stopping time τ, it is natural to try to examine τ using the technology of martingale theory, such as the optional stopping or martingale convergence theorems. This is the basic ideology of the *semimartingale method*. This approach came after Foster; fundamental work was done by J. Lamperti in the 1960s.

What Is a Semimartingale?

The phrase 'semimartingale' has a quite precise technical meaning in stochastic analysis, as well as being an obsolete term for submartingale. For us, it has a much looser meaning: a semimartingale is a process that satisfies some drift condition like $(\star)$, typically only locally. Often, with the aid of a stopping time, one can convert such a process into a true supermartingale or submartingale, as in the case of $(\star)$, but we do not make this demand on all our semimartingales.

Features of This Book

As mentioned above, we use the terminology 'Lyapunov function' in a general sense, and in our usage the term does not presuppose any particular property (stability or otherwise) for the transformed process. When presented with a Foster–Lyapunov result demanding verification of a drift condition such as (⋆), one immediately is faced with the problem of how to choose the Lyapunov function f. Usually there is no fixed rule about how to discover the 'right' Lyapunov function, which must somehow encapsulate one's intuition about the process ξ_n under consideration. For this reason, in this book we not only present the 'right' answers, but also do our best to explain the intuition behind the choice of Lyapunov function in the context of diverse applications and examples.

Finding a suitable Lyapunov function is not always easy, and there is no exact algorithm for that. Nevertheless, there are several intuitive rules and non-rigorous ideas that may help; in the text, we emphasize this kind of heuristic in the following way:

i *Just try a good-looking Lyapunov function, work hard to perform all the computations, and hope for the best.*

We usually avoid adorning the main text with citations to the literature, and instead collect bibliographical notes at the end of each chapter. We have endeavoured to track down original references where possible, and have uncovered several important works of which we were previously unaware; we apologise in advance for any egregious omissions that remain.

Overview of Content

The material is presented in a logical order, but the book has several entry points for the reader. Chapter 1 serves as a gentle introduction to the main theme of the book (non-homogeneous random walks). Chapters 2 and 3 introduce the technical apparatus of the semimartingale approach and describe its application to near-critical processes on the half-line. Our intention is that these two chapters will serve as a useful reference for researchers who wish to use these tools, and so we have tried to give relatively strong versions of the results in some generality. As an antidote to the technical demands of these two chapters, we have included many examples, as our intention is also that Chapters 2 and 3 should prove instructive to the student who wishes to develop an intuition for the method. Chapters 4–6 present applications of the Lyapunov function method to some near-critical stochastic processes. Thus, while these chapters frequently refer to results from Chapter 2 (and Chapter 4 also relies heavily on Chapter 3), our intention is that the reader who is so inclined can take the technical tools for granted and read each of these later chapters as a stand-alone exposition. The final chapter, Chapter 7, switches focus to continuous time; while the development parallels some of the ideas in Chapter 2, this chapter is essentially self-contained.

The section headings in the table of contents provide an indication of the subject matter of each chapter; here we outline briefly what each chapter contains.

Chapter 1 Introduction

This chapter motivates the developments that follow by way of a classical and fundamental model in probability theory: the d-dimensional random walk. We describe the transition from (classical) homogeneous random walk to spatially non-homogeneous random walks, and how the investigation of such models is motivated by theoretical questions arising from trying to go beyond the classical setting and to probe the recurrence/transience phase transition. Immediately the relaxation of spatial homogeneity requires a significant readjustment of random walk intuition: one can readily construct two-dimensional, zero-drift, bounded-jump random walks that are *transient*, for example, provided spatial homogeneity is not enforced, completely contrary to classical behaviour.

Chapter 2 Semimartingale Approach and Markov Chains

This section presents the basic technical apparatus that we rely on for the rest of the book. We review some basic martingale ideas (Doob's inequality, martingale convergence, and the optional stopping theorem) and present a variety of semimartingale tools, including maximal inequalities, results on finiteness of hitting times, and existence and non-existence of moments for hitting times. These results include Foster–Lyapunov criteria for Markov chains, whereby a suitable Lyapunov function enables one to conclude that the process is transient, recurrent, positive recurrent, etc. We provide many examples of the application of these results.

Chapter 3 Lamperti's Problem

This chapter presents applications of the semimartingale tools of Chapter 2 in the context of one-dimensional adapted processes with asymptotically vanishing drift, the so-called *Lamperti's problem*. Lamperti's problem serves as a first important example of a near-critical stochastic process, and is also motivated by its ubiquity arising from the application of the Lyapunov function method to near-critical processes in higher dimensions. This chapter studies in turn various aspects of the asymptotic behaviour of Lamperti processes, including the recurrence classification for processes with asymptotically zero drifts, results on existence and non-existence of passage-time moments, Gamma-type weak convergence results, and almost-sure bounds on the trajectory of the process.

Chapter 4 Many-Dimensional Random Walks

This chapter presents applications of the results of Chapter 3 to many dimensional random walks. The applications in this chapter (and later on) proceed by the Lyapunov function ideology: use a suitably chosen Lyapunov function of the many-dimensional Markov process to obtain a (probably non-Markov) stochastic process in one dimension, which fits into the framework of Chapter 3. We consider in detail the recurrence classification problem for non-homogeneous random walks, with emphasis on the possibility of *anomalous recurrence* behaviour. We also give results on *angular asymptotics* and on the *range* of many-dimensional martingales.

Chapter 5 Heavy Tails

The processes considered in Chapters 3 and 4 are all assumed to have at least one or two moments for their increments. This chapter turns to the *heavy-tailed* case when the first or second increment moment is infinite. We present results for real-valued Markov chains with heavy-tailed jumps, focusing on recurrence classification; we demonstrate how the Lyapunov function approach is equally effective in this heavy-tailed setting.

Chapter 6 Further Applications

This chapter presents a selection of applications of the Lyapunov function method to some near-critical stochastic systems. We consider Markov processes in random environments, some models of interacting particle systems, and a stochastic billiards process. This chapter focuses on recurrence and transience results, obtained by applications of the Foster–Lyapunov criteria from Chapter 2.

Chapter 7 Markov Chains in Continuous Time

For the final chapter of the book, we switch from discrete to continuous time. This chapter presents recent developments on semimartingale techniques for continuous-time discrete-space Markov chains, non-homogeneous both in space and in jump rates. For example, the (embedded) jump process might be of Lamperti type, while the rates are *not* uniformly bounded away from

0 and ∞. This gives rise to additional phenomena over the discrete-time setting. We present conditions for existence of moments of hitting times in this continuous-time setting, and give criteria for explosion and implosion for such processes, again using semimartingale techniques.

Acknowledgements

We express our immense gratitude to all our co-authors. Especial thanks are appropriate here to those whose collaborations directly relate to material presented in one or more of the chapters of this book: Sanjar Aspandiiarov, Inna Asymont, Vladimir Belitsky, Francis Comets, Guy Fayolle, Pablo Ferrari, Nicholas Georgiou, Ostap Hryniv, Roudolf Iasnogorodski, Iain MacPhee, Vadim Malyshev, Aleksandar Mijatović, Yuval Peres, Dimitri Pétritis, Perla Sousi, Marina Vachkovskaia, and Stanislav Volkov; their ideas permeate this book.

We are grateful to Nicholas Georgiou for providing Figures 4.1, 4.2, and 4.4. Parts of the manuscript were read by Marcelo Costa, Nicholas Georgiou, Chak Hei Lo, and James McRedmond; we are grateful for their comments and suggestions.

This project was started when AW was at the University of Strathclyde; AW also acknowledges support during part of this project by the Engineering and Physical Sciences Research Council [grant number EP/J021784/1].

Notation

Miscellaneous

We write $a := \cdots$ to indicate the definition of a. Occasionally we also use $\cdots =: a$. We use the standard abbreviation 'a.s.' for 'almost surely' (with probability 1).

Sets, Probabilities, and Events

The set of integers is $\mathbb{Z}$. The natural numbers are $\mathbb{N} := \{1, 2, 3, \ldots\}$. The non-negative integers are $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. The real numbers are $\mathbb{R}$. The non-negative half-line is $\mathbb{R}_+ := [0, \infty)$. It is often convenient to extend these sets to include infinities, so we set $\overline{\mathbb{Z}}_+ := \mathbb{Z}_+ \cup \{+\infty\}$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, and $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$. For a set S, we write $\#S$ for its cardinality (number of elements, when finite). For a measurable subset A of $\mathbb{R}^d$, we write $|A|$ for its d-dimensional Lebesgue measure (volume).

We will always assume an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$; expectation with respect to $\mathbb{P}$ will be denoted $\mathbb{E}$. For $A \subseteq \Omega$, we denote its complement by $A^{\mathrm{c}} = \Omega \setminus A$.

If $(A_n, n \in \mathbb{Z}_+)$ is a sequence of events, we use the standard notation

$$\begin{aligned}
\{A_n \text{ i.o.}\} &= \{A_n \text{ infinitely often}\} \\
&= \limsup A_n = \cap_{m\geq 0} \cup_{n \geq m} A_n; \\
\{A_n \text{ ev.}\} &= \{A_n \text{ eventually}\} = \{A_n \text{ all but finitely often}\} \\
&= \liminf A_n = \cup_{m\geq 0} \cap_{n \geq m} A_n; \\
\{A_n \text{ f.o.}\} &= \{A_n \text{ finitely often}\} = \{A_n \text{ i.o.}\}^{\mathrm{c}} = \{A_n^{\mathrm{c}} \text{ ev.}\} \\
&= \cup_{m\geq 0} \cap_{n \geq m} A_n^{\mathrm{c}}.
\end{aligned}$$

Conventions and Empty Evaluations

Unless otherwise stated, the following conventions are in force throughout. An empty sum is zero, and an empty product is one. Also $\inf \emptyset := +\infty$, and $\sup \emptyset := 0$.

Real Numbers, Vectors, and Matrices

For x a real number we set

$$x^+ := \max\{0, x\}, \quad \text{and} \quad x^- := -\min\{0, x\},$$

so $x = x^+ - x^-$ and $|x| = x^+ + x^-$. For real numbers x and y, we set

$$x \wedge y := \min\{x, y\}, \quad \text{and} \quad x \vee y := \max\{x, y\}.$$

For $x \in \mathbb{R}$ we write $\lfloor x \rfloor$ for the largest integer not exceeding x, and $\lceil x \rceil$ for the smallest integer no less than x; so $\lceil x \rceil - \lfloor x \rfloor \in \{0, 1\}$, and is 0 if and only if $x \in \mathbb{Z}$.

For emphasis, we sometimes denote monotone convergence by '$\uparrow$' and '$\downarrow$', so for a sequence $a_x \in \mathbb{R}$, $a_x \uparrow a$ as $x \to \infty$ means that $\lim_{x\to\infty} a_x = a$ and $a_x \leq a_y$ for all $x \leq y$.

For a matrix M with real-valued entries we write $M^\top$ for its transpose, and $\lambda_{\max}(M)$ for its maximum eigenvalue. We usually use boldface letters for vectors in $\mathbb{R}^d$, and write, for example, $\mathbf{x} = (x_1, \ldots, x_d)^\top$ in Cartesian components; for definiteness, vectors $\mathbf{x} \in \mathbb{R}^d$ are viewed as column vectors throughout. The origin in $\mathbb{R}^d$ is denoted by $\mathbf{0}$. We write $\| \cdot \|$ for the Euclidean norm on $\mathbb{R}^d$. We write $\mathbf{e}_1, \ldots, \mathbf{e}_d$ for the standard orthonormal basis of $\mathbb{R}^d$, and for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ we denote their scalar product by $\mathbf{u}^\top \mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$, or $\langle \mathbf{u}, \mathbf{v} \rangle$. For a non-zero vector $\mathbf{x} \in \mathbb{R}^d$ we write $\hat{\mathbf{x}} := \mathbf{x}/\|\mathbf{x}\|$ for the corresponding unit vector, and we adopt the convention $\hat{\mathbf{0}} := \mathbf{0}$. The unit-radius sphere in $\mathbb{R}^d$ is

$$\mathbb{S}^{d-1} := \{\mathbf{u} \in \mathbb{R}^d : \|\mathbf{u}\| = 1\}.$$

The (closed) Euclidean d-ball centred at $\mathbf{x} \in \mathbb{R}^d$ with radius $r \in \mathbb{R}_+$ is

$$B(\mathbf{x}; r) := \{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{y}\| \leq r\}.$$

We denote the d by d identity matrix by I_d. For a square matrix M with real-valued entries, we write $\operatorname{tr} M$ for its trace. A d by d real matrix M acts on column vectors $\mathbf{x} \in \mathbb{R}^d$ as an affine function from $\mathbb{R}^d$ to $\mathbb{R}^d$ via $\mathbf{x} \mapsto M\mathbf{x}$. The associated matrix (operator) norm $\| \cdot \|_{\text{op}}$ induced by the Euclidean norm on $\mathbb{R}^d$ is

$$\|M\|_{\text{op}} := \sup_{\mathbf{u} \in \mathbb{S}^{d-1}} \|M\mathbf{u}\|.$$

Using the variational characterization of the largest eigenvalue as $\lambda_{\max}(M) = \sup_{\mathbf{u}\in\mathbb{S}^{d-1}}(\mathbf{u}^\top M\mathbf{u})$, we note that

$$\sup_{\mathbf{u}\in\mathbb{S}^{d-1}} \|M\mathbf{u}\|^2 = \sup_{\mathbf{u}\in\mathbb{S}^{d-1}} \big(\mathbf{u}^\top M^\top M\mathbf{u}\big) = \lambda_{\max}(M^\top M),$$

so that an alternative expression for the operator norm is

$$\|M\|_{\rm op} = \big(\lambda_{\max}(M^\top M)\big)^{1/2}.$$

Functions

The natural logarithm of x is $\log x$. For $r \in \mathbb{R}$, we write $\log^r x$ for $(\log x)^r$. We also write $\log_1 x := \log x$, and for $k \geq 2$ set $\log_k x := \log \log_{k-1} x$, so that $\log_k x$ is the k-fold iterated logarithm of x.

Random Variables

We use $\mathbf{1}$ for the indicator function of an event, indicated either in curly braces as $\mathbf{1}\{\,\cdot\,\}$ or via a previously assigned symbol such as $\mathbf{1}(A)$.

For a sub-σ-field $\mathcal{G}$ of $\mathcal{F}$, $\mathcal{G}$-measurable random variables X and Y, and an event $A \in \mathcal{G}$, the statement '$X = Y$ on A' is equivalent to '$X\mathbf{1}(A) = Y\mathbf{1}(A)$'; similarly for inequalities.

We use the standard notation for essential supremum and infimum: for a real-valued random variable X,

$$\operatorname{ess\,inf} X := \sup\{x \in \mathbb{R} : \mathbb{P}[X \geq x] = 1\};$$
$$\operatorname{ess\,sup} X := \inf\{x \in \mathbb{R} : \mathbb{P}[X > x] = 0\}.$$

For $\mathbb{R}^d$-valued random variables $X, X_1, X_2, \ldots$, we denote convergence by $X_n \to X$ qualified by the mode of convergence in the text; for example, $X_n \to X$ a.s. means that $\mathbb{P}[X_n \to X] = 1$. Sometimes for compactness we write '$\xrightarrow{\text{a.s.}}$', '$\xrightarrow{p}$', and '$\xrightarrow{d}$' for convergence almost surely, in probability, and in distribution, respectively.

Asymptotics

We reserve unadorned Landau $O(\,\cdot\,)$ and $o(\,\cdot\,)$ symbols for the case where implicit constants are *non-random*. Thus for a real-valued function f and an $\mathbb{R}_+$-valued function g, the expression $f(x) = O(g(x))$ means that there exist finite deterministic constants C and x_0 such that $|f(x)| \leq Cg(x)$ for all $x \geq x_0$. Similarly, $f(x) = o(g(x))$ means that for any $\varepsilon > 0$ there exists a finite deterministic x_ε such that $|f(x)| \leq \varepsilon g(x)$ for all $x \geq x_\varepsilon$.

Cambridge University Press
978-1-107-02669-8 — Non-homogeneous Random Walks
Mikhail Menshikov , Serguei Popov , Andrew Wade
Frontmatter
More Information

It is convenient to extend the O, o notation to permit random variables, but it is important to do this carefully to avoid ambiguous formulas. Given a σ-field $\mathcal{F}$ and $\mathcal{F}$-measurable random variables X and Y, we write $O_X^{\mathcal{F}}(Y)$ to represent an $\mathcal{F}$-measurable random variable such that there exist finite deterministic constants C and x_0 such that $|O_X^{\mathcal{F}}(Y)| \leq CY$ on the event $\{X \geq x_0\}$. Although this notation is a little cumbersome, we feel the extra clarity is worthwhile, as an ambiguous $O(\,\cdot\,)$ can hide a multitude of sins.