

GEOMETRY OF QUANTUM STATES

An Introduction to Quantum Entanglement

Quantum information theory is a branch of science at the frontiers of physics, mathematics and information science, and offers a variety of solutions that are impossible using classical theory. This book provides a detailed introduction to the key concepts used in processing quantum information and reveals that quantum mechanics is a generalisation of classical probability theory.

The second edition contains new sections and entirely new chapters: the hot topic of multipartite entanglement; in-depth discussion of the discrete structures in finite dimensional Hilbert space, including unitary operator bases, mutually unbiased bases, symmetric informationally complete generalised measurements, discrete Wigner functions and unitary designs; the Gleason and Kochen–Specker theorems; the proof of the Lieb conjecture; the measure concentration phenomenon; and the Hastings’ non-additivity theorem.

This richly-illustrated book will be useful to a broad audience of graduates and researchers interested in quantum information theory. Exercises follow each chapter, with hints and answers supplied.

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SECOND EDITION

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Contents

<i>Preface</i>	<i>page</i> xi
<i>Acknowledgements</i>	xv
1 Convexity, colours and statistics	1
1.1 Convex sets	1
1.2 High dimensional geometry	9
1.3 Colour theory	15
1.4 What is ‘distance’?	18
1.5 Probability and statistics	25
Problems	28
2 Geometry of probability distributions	29
2.1 Majorisation and partial order	29
2.2 Shannon entropy	35
2.3 Relative entropy	41
2.4 Continuous distributions and measures	46
2.5 Statistical geometry and the Fisher–Rao metric	48
2.6 Classical ensembles	55
2.7 Generalised entropies	56
Problems	62
3 Much ado about spheres	63
3.1 Spheres	63
3.2 Parallel transport and statistical geometry	68
3.3 Complex, Hermitian, and Kähler manifolds	74
3.4 Symplectic manifolds	80
3.5 The Hopf fibration of the 3-sphere	82
3.6 Fibre bundles and their connections	88
3.7 The 3-sphere as a group	95

vi	<i>Contents</i>	
	3.8 Cosets and all that	99
	Problems	101
4	Complex projective spaces	103
	4.1 From art to mathematics	103
	4.2 Complex projective geometry	108
	4.3 Complex curves, quadrics and the Segre embedding	110
	4.4 Stars, spinors and complex curves	114
	4.5 The Fubini–Study metric	117
	4.6 \mathbb{CP}^n illustrated	122
	4.7 Symplectic geometry and the Fubini–Study measure	129
	4.8 Fibre bundle aspects	131
	4.9 Grassmannians and flag manifolds	133
	Problems	137
5	Outline of quantum mechanics	138
	5.1 Quantum mechanics	138
	5.2 Qubits and Bloch spheres	140
	5.3 The statistical and the Fubini–Study distances	143
	5.4 A real look at quantum dynamics	146
	5.5 Time reversals	150
	5.6 Classical and quantum states: a unified approach	154
	5.7 Gleason and Kochen–Specker	157
	Problems	164
6	Coherent states and group actions	165
	6.1 Canonical coherent states	165
	6.2 Quasi-probability distributions on the plane	170
	6.3 Bloch coherent states	178
	6.4 From complex curves to $SU(K)$ coherent states	184
	6.5 $SU(3)$ coherent states	186
	Problems	190
7	The stellar representation	191
	7.1 The stellar representation in quantum mechanics	191
	7.2 Orbits and coherent states	194
	7.3 The Husimi function	197
	7.4 Wehrl entropy and the Lieb conjecture	202
	7.5 Generalised Wehrl entropies	205
	7.6 Random pure states	207
	7.7 From the transport problem to the Monge distance	214
	Problems	217

Contents

vii

8	The space of density matrices	219
8.1	Hilbert–Schmidt space and positive operators	220
8.2	The set of mixed states	223
8.3	Unitary transformations	226
8.4	The space of density matrices as a convex set	229
8.5	Stratification	234
8.6	Projections and cross-sections	239
8.7	An algebraic afterthought	244
8.8	Summary	247
	Problems	248
9	Purification of mixed quantum states	249
9.1	Tensor products and state reduction	250
9.2	The Schmidt decomposition	252
9.3	State purification and the Hilbert–Schmidt bundle	255
9.4	A first look at the Bures metric	258
9.5	Bures geometry for $N = 2$	261
9.6	Further properties of the Bures metric	263
	Problems	266
10	Quantum operations	267
10.1	Measurements and POVMs	267
10.2	Algebraic detour: matrix reshaping and reshuffling	274
10.3	Positive and completely positive maps	278
10.4	Environmental representations	283
10.5	Some spectral properties	286
10.6	Unital and bistochastic maps	288
10.7	One qubit maps	291
	Problems	295
11	Duality: maps versus states	296
11.1	Positive and decomposable maps	296
11.2	Dual cones and super-positive maps	304
11.3	Jamiołkowski isomorphism	305
11.4	Quantum maps and quantum states	309
	Problems	311
12	Discrete structures in Hilbert space	313
12.1	Unitary operator bases and the Heisenberg groups	313
12.2	Prime, composite and prime power dimensions	317
12.3	More unitary operator bases	322

12.4	Mutually unbiased bases	326
12.5	Finite geometries and discrete Wigner functions	333
12.6	Clifford groups and stabilizer states	338
12.7	Some designs	343
12.8	SICs	347
	Problems	353
13	Density matrices and entropies	355
13.1	Ordering operators	356
13.2	Von Neumann entropy	359
13.3	Quantum relative entropy	365
13.4	Other entropies	369
13.5	Majorisation of density matrices	372
13.6	Proof of the Lieb conjecture	376
13.7	Entropy dynamics	380
	Problems	384
14	Distinguishability measures	386
14.1	Classical distinguishability measures	386
14.2	Quantum distinguishability measures	391
14.3	Fidelity and statistical distance	396
	Problems	401
15	Monotone metrics and measures	402
15.1	Monotone metrics	402
15.2	Product measures and flag manifolds	408
15.3	Hilbert–Schmidt measure	410
15.4	Bures measure	414
15.5	Induced measures	416
15.6	Random density matrices	418
15.7	Random operations	422
15.8	Concentration of measure	426
	Problems	431
16	Quantum entanglement	433
16.1	Introducing entanglement	433
16.2	Two-qubit pure states: entanglement illustrated	436
16.3	Maximally entangled states	441
16.4	Pure states of a bipartite system	444
16.5	A first look at entangled mixed states	454
16.6	Separability criteria	460

<i>Contents</i>	ix
16.7 Geometry of the set of separable states	466
16.8 Entanglement measures	471
16.9 Two-qubit mixed states	481
Problems	491
17 Multipartite entanglement	493
17.1 How much is three larger than two?	493
17.2 Botany of states	494
17.3 Permutation symmetric states	498
17.4 Invariant theory and quantum states	503
17.5 Monogamy relations and global multipartite entanglement	515
17.6 Local spectra and the momentum map	518
17.7 AME states and error-correcting codes	525
17.8 Entanglement in quantum spin systems	532
Problems	542
<i>Epilogue</i>	544
<i>Appendix A</i> Basic notions of differential geometry	545
A.1 Differential forms	545
A.2 Riemannian curvature	546
A.3 A key fact about mappings	547
<i>Appendix B</i> Basic notions of group theory	549
B.1 Lie groups and Lie algebras	549
B.2 $SU(2)$	550
B.3 $SU(N)$	551
B.4 Homomorphisms between low-dimensional groups	552
<i>Appendix C</i> Geometry: do it yourself	553
<i>Appendix D</i> Hints and answers to the exercises	557
<i>References</i>	572
<i>Index</i>	614

Preface

Preface to the first edition

The geometry of quantum states is a highly interesting subject in itself. It is also relevant in view of possible applications in the rapidly developing fields of quantum information and quantum computing.

But what is it? In physics words like ‘states’ and ‘system’ are often used. Skipping lightly past the question of what these words mean – it will be made clear by practice – it is natural to ask for the properties of the space of all possible states of a given system. The simplest state space occurs in computer science: a ‘bit’ has a space of states that consists simply of two points, representing on and off. In probability theory the state space of a bit is really a line segment, since the bit may be ‘on’ with some probability between zero and one. In general the state spaces used in probability theory are ‘convex hulls’ of a discrete or continuous set of points. The geometry of these simple state spaces is surprisingly subtle – especially since different ways of distinguishing probability distributions give rise to different notions of distance, each with their own distinct operational meaning. There is an old idea saying that a geometry can be understood once it is understood what linear transformations are acting on it, and we will see that this is true here as well.

The state spaces of classical mechanics are – at least from the point of view that we adopt – just instances of the state spaces of classical probability theory, with the added requirement that the sample spaces (whose ‘convex hull’ we study) are large enough, and structured enough, so that the transformations acting on them include canonical transformations generated by Hamiltonian functions.

In quantum theory the distinction between probability theory and mechanics goes away. The simplest quantum state space is these days known as a ‘qubit’. There are many physical realisations of a qubit, from silver atoms of spin $1/2$ (assuming that we agree to measure only their spin) to the qubits that are literally designed in today’s laboratories. As a state space a qubit is a three-dimensional ball; each diameter of the ball is the state space of some classical bit, and there are so many

bits that their sample spaces conspire to form a space – namely the surface of the ball – large enough to carry the canonical transformations that are characteristic of mechanics. Hence the word quantum mechanics.

It is not particularly difficult to understand a three-dimensional ball, or to see how this description emerges from the usual description of a qubit in terms of a complex two-dimensional Hilbert space. In this case we can take the word *geometry* literally – there will exist a one-to-one correspondence between pure states of the qubit and the points of the surface of the Earth. Moreover, at least as far as the surface is concerned, its geometry has a statistical meaning when transcribed to the qubit (although we will see some strange things happening in the interior).

As the dimension of the Hilbert space goes up, the geometry of the state spaces becomes very intricate, and qualitatively new features arise – such as the subtle way in which composite quantum systems are represented. Our purpose is to describe this geometry. We believe it is worth doing. Quantum state spaces are more wonderful than classical state spaces, and in the end composite systems of qubits may turn out to have more practical applications than the bits themselves already have.

A few words about the contents of our book. As a glance at the Contents will show, there are 17 chapters, culminating in a long chapter on ‘entanglement’. Along the way, we take our time to explore many curious byways of geometry. We expect that you – the reader – are familiar with the principles of quantum mechanics at the advanced undergraduate level. We do not really expect more than that, and should you be unfamiliar with quantum mechanics we hope that you will find some sections of the book profitable anyway. You may start reading any chapter – if you find it incomprehensible we hope that the cross-references and the index will enable you to see what parts of the earlier chapters may be helpful to you. In the unlikely event that you are not even interested in quantum mechanics, you may perhaps enjoy our explanations of some of the geometrical ideas that we come across.

Of course there are limits to how independent the chapters can be of each other. Convex set theory (Chapter 1) pervades all statistical theories, and hence all our chapters. The ideas behind the classical Shannon entropy and the Fisher–Rao geometry (Chapter 2) must be brought in to explain quantum mechanical entropies (Chapter 12) and quantum statistical geometry (Chapters 9 and 13). Sometimes we have to assume a little extra knowledge on the part of the reader, but since no chapter in our book assumes that all the previous chapters have been understood, this should not pose any great difficulties.

We have made a special effort to illustrate the geometry of quantum mechanics. This is not always easy, since the spaces that we encounter more often than not have a dimension higher than three. We have simply done the best we could. To facilitate self-study each chapter concludes with problems for the reader, while some additional geometrical exercises are presented in the final appendix.

Once and for all, let us say that we limit ourselves to finite-dimensional state spaces. We do this for two reasons. One of them is that it simplifies the story very much, and the other is that finite-dimensional systems are of great independent interest in real experiments.

The entire book may be considered as an introduction to quantum entanglement. This very non-classical feature provides a key resource for several modern applications of quantum mechanics including quantum cryptography, quantum computing and quantum communication. We hope that our book may be useful for graduate and postgraduate students of physics. It is written first of all for readers who do not read the mathematical literature everyday, but we hope that students of mathematics and of the information sciences will find it useful as well, since they also may wish to learn about quantum entanglement.

We have been working on the book for about five years. Throughout this time we enjoyed the support of Stockholm University, the Jagiellonian University in Cracow, and the Center for Theoretical Physics of the Polish Academy of Sciences in Warsaw. The book was completed in Waterloo during our stay at the Perimeter Institute for Theoretical Physics. The motto at its main entrance – ΑΣΠΟΥΔΑΣΤΟΣ ΠΕΡΙ ΓΕΩΜΕΤΡΙΑΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ – proved to be a lucky omen indeed, and we are pleased to thank the Institute for creating optimal working conditions for us, and to thank all the knowledgeable colleagues working there for their help, advice, and support.

We are grateful to Erik Aurell for his commitment to Polish–Swedish collaboration; without him the book would never have been started. It is a pleasure to thank our colleagues with whom we have worked on related projects: Johan Brännlund, Åsa Ericsson, Sven Gnutzmann, Marek Kuś, Florian Mintert, Magdalena Sinołowska, Hans-Jürgen Sommers and Wojciech Słomczyński. We are grateful to them, and to many others who helped us to improve the manuscript. If it never reached perfection, it was our fault, not theirs. Let us mention some of the others: Robert Alicki, Anders Bengtsson, Iwo Białynicki-Birula, Rafał Demkowicz-Dobrzański, Johan Grundberg, Sören Holst, Göran Lindblad, and Marcin Musz. We have also enjoyed constructive interactions with Matthias Christandl, Jens Eisert, Peter Harremoës, Michał, Paweł and Ryszard Horodeccy, Vivien Kendon, Artur Łoziński, Christian Schaffner, Paul Slater, and William Wootters.

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Waterloo, 12 March 2005

*Ingemar Bengtsson
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Preface to the second edition

More than a decade has passed since we completed the first edition of the book. Much has happened during it. We have not tried to take all recent valuable contributions into account, but some of them can be found here. The Lieb conjecture for spin coherent states has been proved by Lieb and Solovej, and an important additivity conjecture for quantum channels has been disproved. Since these conjectures were discussed at some length in the first edition we felt we had to say more about them. However, our main concern with this second edition has been to improve explanations and to remove mistakes (while trying not to introduce new ones).

There are two new chapters: one of them is centred around the finite Weyl–Heisenberg group, and the discrete structures it gives rise to. The other tries to survey the vast field of multipartite entanglement, which was relegated to a footnote or two in the first edition. There are also a few new sections: They concern Gleason’s theorem and quantum contextuality, cross-sections and projections of the body of mixed quantum states, and the concentration of measure in high dimensions (which is needed to understand what is now Hastings’ non-additivity theorem).

We are grateful to several people who helped us with the work on this edition. It is a pleasure to thank Felix Huber, Pedro Lamberti and Marcus Müller for their comments on the first edition. We are indebted to Radosław Adamczak, Ole Andersson, Marcus Appleby, Adán Cabello, Runyao Duan, Shmuel Friedland, Dardo Goyeneche, David Gross, Michał and Paweł Horodeccy, Ted Jacobson, David Jennings, Marek Kuś, Ion Nechita, Zbigniew Puchała, Wojciech Roga, Adam Sawicki, Stanisław Szarek, Stephan Weis, Andreas Winter, Iwona Wintrowicz, and Huangjun Zhu, for reading some fragments of the text and providing us with valuable remarks. Had they read all of it, we are sure that it would have been perfect! We thank Kate Blanchfield, Piotr Gawron, Lia Pugliese, Konrad Szymański, and Maria Życzkowska, for preparing for us two-dimensional figures and three-dimensional printouts, models and photos for the book.

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Toruń, 11 June 2016

*Ingemar Bengtsson
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