GEOMETRY OF QUANTUM STATES

An Introduction to Quantum Entanglement

Quantum information theory is a branch of science at the frontiers of physics, mathematics and information science, and offers a variety of solutions that are impossible using classical theory. This book provides a detailed introduction to the key concepts used in processing quantum information and reveals that quantum mechanics is a generalisation of classical probability theory.

The second edition contains new sections and entirely new chapters: the hot topic of multipartite entanglement; in-depth discussion of the discrete structures in finite dimensional Hilbert space, including unitary operator bases, mutually unbiased bases, symmetric informationally complete generalised measurements, discrete Wigner functions and unitary designs; the Gleason and Kochen–Specker theorems; the proof of the Lieb conjecture; the measure concentration phenomenon; and the Hastings' non-additivity theorem.

This richly-illustrated book will be useful to a broad audience of graduates and researchers interested in quantum information theory. Exercises follow each chapter, with hints and answers supplied.

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SECOND EDITION

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Preface

Preface to the first edition

The geometry of quantum states is a highly interesting subject in itself. It is also relevant in view of possible applications in the rapidly developing fields of quantum information and quantum computing.

But what is it? In physics words like 'states' and 'system' are often used. Skipping lightly past the question of what these words mean – it will be made clear by practice – it is natural to ask for the properties of the space of all possible states of a given system. The simplest state space occurs in computer science: a 'bit' has a space of states that consists simply of two points, representing on and off. In probability theory the state space of a bit is really a line segment, since the bit may be 'on' with some probability between zero and one. In general the state spaces used in probability theory are 'convex hulls' of a discrete or continuous set of points. The geometry of these simple state spaces is surprisingly subtle – especially since different ways of distinguishing probability distributions give rise to different notions of distance, each with their own distinct operational meaning. There is an old idea saying that a geometry can be understood once it is understood what linear transformations are acting on it, and we will see that this is true here as well.

The state spaces of classical mechanics are – at least from the point of view that we adopt – just instances of the state spaces of classical probability theory, with the added requirement that the sample spaces (whose 'convex hull' we study) are large enough, and structured enough, so that the transformations acting on them include canonical transformations generated by Hamiltonian functions.

In quantum theory the distinction between probability theory and mechanics goes away. The simplest quantum state space is these days known as a 'qubit'. There are many physical realisations of a qubit, from silver atoms of spin 1/2 (assuming that we agree to measure only their spin) to the qubits that are literally designed in today's laboratories. As a state space a qubit is a three–dimensional ball; each diameter of the ball is the state space of some classical bit, and there are so many bits that their sample spaces conspire to form a space – namely the surface of the ball – large enough to carry the canonical transformations that are characteristic of mechanics. Hence the word quantum mechanics.

It is not particularly difficult to understand a three–dimensional ball, or to see how this description emerges from the usual description of a qubit in terms of a complex two-dimensional Hilbert space. In this case we can take the word *geometry* literally – there will exist a one-to-one correspondence between pure states of the qubit and the points of the surface of the Earth. Moreover, at least as far as the surface is concerned, its geometry has a statistical meaning when transcribed to the qubit (although we will see some strange things happening in the interior).

As the dimension of the Hilbert space goes up, the geometry of the state spaces becomes very intricate, and qualitatively new features arise – such as the subtle way in which composite quantum systems are represented. Our purpose is to describe this geometry. We believe it is worth doing. Quantum state spaces are more wonderful than classical state spaces, and in the end composite systems of qubits may turn out to have more practical applications than the bits themselves already have.

A few words about the contents of our book. As a glance at the Contents will show, there are 17 chapters, culminating in a long chapter on 'entanglement'. Along the way, we take our time to explore many curious byways of geometry. We expect that you – the reader – are familiar with the principles of quantum mechanics at the advanced undergraduate level. We do not really expect more than that, and should you be unfamiliar with quantum mechanics we hope that you will find some sections of the book profitable anyway. You may start reading any chapter – if you find it incomprehensible we hope that the cross-references and the index will enable you to see what parts of the earlier chapters may be helpful to you. In the unlikely event that you are not even interested in quantum mechanics, you may perhaps enjoy our explanations of some of the geometrical ideas that we come across.

Of course there are limits to how independent the chapters can be of each other. Convex set theory (Chapter 1) pervades all statistical theories, and hence all our chapters. The ideas behind the classical Shannon entropy and the Fisher–Rao geometry (Chapter 2) must be brought in to explain quantum mechanical entropies (Chapter 12) and quantum statistical geometry (Chapters 9 and 13). Sometimes we have to assume a little extra knowledge on the part of the reader, but since no chapter in our book assumes that all the previous chapters have been understood, this should not pose any great difficulties.

We have made a special effort to illustrate the geometry of quantum mechanics. This is not always easy, since the spaces that we encounter more often than not have a dimension higher than three. We have simply done the best we could. To facilitate self-study each chapter concludes with problems for the reader, while some additional geometrical exercises are presented in the final appendix.

Once and for all, let us say that we limit ourselves to finite-dimensional state spaces. We do this for two reasons. One of them is that it simplifies the story very much, and the other is that finite-dimensional systems are of great independent interest in real experiments.

The entire book may be considered as an introduction to quantum entanglement. This very non-classical feature provides a key resource for several modern applications of quantum mechanics including quantum cryptography, quantum computing and quantum communication. We hope that our book may be useful for graduate and postgraduate students of physics. It is written first of all for readers who do not read the mathematical literature everyday, but we hope that students of mathematics and of the information sciences will find it useful as well, since they also may wish to learn about quantum entanglement.

We have been working on the book for about five years. Throughout this time we enjoyed the support of Stockholm University, the Jagiellonian University in Cracow, and the Center for Theoretical Physics of the Polish Academy of Sciences in Warsaw. The book was completed in Waterloo during our stay at the Perimeter Institute for Theoretical Physics. The motto at its main entrance – ΑΣΠΟΥΔΑΣΤΟΣ ΠΕΡΙ ΓΕΩΜΕΤΡΙΑΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ – proved to be a lucky omen indeed, and we are pleased to thank the Institute for creating optimal working conditions for us, and to thank all the knowledgeable colleagues working there for their help, advice, and support.

We are grateful to Erik Aurell for his commitment to Polish–Swedish collaboration; without him the book would never have been started. It is a pleasure to thank our colleagues with whom we have worked on related projects: Johan Brännlund, Åsa Ericsson, Sven Gnutzmann, Marek Kuś, Florian Mintert, Magdalena Sinołęcka, Hans-Jürgen Sommers and Wojciech Słomczyński. We are grateful to them, and to many others who helped us to improve the manuscript. If it never reached perfection, it was our fault, not theirs. Let us mention some of the others: Robert Alicki, Anders Bengtsson, Iwo Białynicki-Birula, Rafał Demkowicz-Dobrzański, Johan Grundberg, Sören Holst, Göran Lindblad, and Marcin Musz. We have also enjoyed constructive interactions with Matthias Christandl, Jens Eisert, Peter Harremoës, Michał, Paweł and Ryszard Horodeccy, Vivien Kendon, Artur Łoziński, Christian Schaffner, Paul Slater, and William Wootters.

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Waterloo, 12 March 2005

Ingemar Bengtsson
Karol Życzkowski

Preface to the second edition

More than a decade has passed since we completed the first edition of the book. Much has happened during it. We have not tried to take all recent valuable contributions into account, but some of them can be found here. The Lieb conjecture for spin coherent states has been proved by Lieb and Solovej, and an important additivity conjecture for quantum channels has been disproved. Since these conjectures were discussed at some length in the first edition we felt we had to say more about them. However, our main concern with this second edition has been to improve explanations and to remove mistakes (while trying not to introduce new ones).

There are two new chapters: one of them is centred around the finite Weyl–Heisenberg group, and the discrete structures it gives rise to. The other tries to survey the vast field of multipartite entanglement, which was relegated to a footnote or two in the first edition. There are also a few new sections: They concern Gleason's theorem and quantum contextuality, cross–sections and projections of the body of mixed quantum states, and the concentration of measure in high dimensions (which is needed to understand what is now Hastings' non-additivity theorem).

We are grateful to several people who helped us with the work on this edition. It is a pleasure to thank Felix Huber, Pedro Lamberti and Marcus Müller for their comments on the first edition. We are indebted to Radosław Adamczak, Ole Andersson, Marcus Appleby, Adán Cabello, Runyao Duan, Shmuel Friedland, Dardo Goyeneche, David Gross, Michał and Paweł Horodeccy, Ted Jacobson, David Jennings, Marek Kuś, Ion Nechita, Zbigniew Puchała, Wojciech Roga, Adam Sawicki, Stanisław Szarek, Stephan Weis, Andreas Winter, Iwona Wintrowicz, and Huangjun Zhu, for reading some fragments of the text and providing us with valuable remarks. Had they read all of it, we are sure that it would have been perfect! We thank Kate Blanchfield, Piotr Gawron, Lia Pugliese, Konrad Szymański, and Maria Życzkowska, for preparing for us two-dimensional figures and three-dimensional printouts, models and photos for the book.

We thank Simon Capelin and his team at Cambridge University Press for the long and fruitful work with us. Most of all we thank Martha and Jolanta for their understanding and infinite patience.

Toruń, 11 June 2016

Ingemar Bengtsson
Karol Życzkowski

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Figures 4.10–13 and 16.1–3 have appeared already in our reference [112], which was published in the *International Journal of Physics* **A17** Issue 31 (2002), coauthored with Johan Brännlund. They appear here with the kind permission of the copyright holder, World Scientific.

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