Introduction

Highest weight modules play a key role in the representation theory of several classes of algebraic objects occurring in Lie theory, including Lie algebras, Lie groups, algebraic groups, Chevalley groups and quantized enveloping algebras. In many of the most important situations, the weights may be regarded as points in Euclidean space, $\mathbb{R}^n$, and there is a finite group (called a Weyl group) that acts on the set of weights by linear transformations. The minuscule representations are those for which the Weyl group acts transitively on the weights, and the highest weight of such a representation is called a minuscule weight. The term "minuscule weight" is a translation of Bourbaki's term *poids minuscule* [8, VIII, section 7.3]; the spelling "miniscule" is also found in the literature, although less commonly, and Russian-speaking authors often call minuscule weights *microweights*. The list of minuscule representations is as follows: all fundamental representations in type A_n, the natural representations in types C_n and D_n, the spin representations in types B_n and D_n, the two 27-dimensional representations in type E_6 and the 56-dimensional representation in type E_7.

Minuscule weights and minuscule representations are important because they occur in a wide variety of contexts in mathematics and physics, especially in representation theory and algebraic geometry. Minuscule representations are the starting point of Standard Monomial Theory developed by Lakshmibai, Seshadri and others [42], and they play a key role in the geometry of Schubert varieties.

One of the advantages of minuscule representations is that they are often much easier to understand than typical representations of the same objects. For example, Seshadri [70] proves that standard monomials give a basis for the homogeneous coordinate ring of G/P in the minuscule case, which is significantly more tractable than the general case. Littelmann's path model for representations of Lie algebras [45] is also much simpler when minuscule representations are involved. Minuscule representations have many other algebraic applications: they are useful in the theory of Chevalley groups, where they are used to construct minuscule weight geometries [5, section 6], and they also appear in the context of Bruhat decompositions [86]. Certain questions in the representation theory of algebraic groups can be reduced to questions about minuscule representations [80]. In the theory of Macdonald polynomials [85, section 6], minuscule weights are used to define the Macdonald operators.

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these topics, and we will refer to these when necessary instead of developing the relevant theory from scratch. For Lie algebras, we recommend the books by Erdmann and Wildon [23] for a beginner, Carter [11] for an intermediate reader, and Kac [37] for an expert. For Weyl groups and Coxeter groups, we recommend the books by Humphreys [36] for a beginner, Björner and Brenti [4] for a reader interested in combinatorics, and Geck and Pfeiffer [26] for a reader interested in computation or representation theory. In contrast, the approach of this book is to develop both Lie algebras and Weyl groups using a single, example driven, combinatorial approach that makes explicit calculations easy. In particular, it should be possible, without a computer, to understand all (or almost all) the examples and exercises in this book.

A reader familiar with linear algebra, groups, rings and point set topology should be able in principle to read this book from cover to cover, treating it as a romp through a circle of algebraic and combinatorial ideas with minuscule representations at the centre. However, the material is arranged to make the book useful as a reference. The reader who wishes to browse is advised to start by looking at the many examples and exercises, particularly those in the last four chapters. Readers already familiar with some of the objects of study may like to try to solve the exercises using their own methods. Each chapter of this book concludes with a section of Notes and references, which includes historical notes, references to the literature and directions for further reading.

The chapters are structured as follows. Chapter 1 introduces Lie algebras and Weyl groups of types A_n, B_n, C_n and D_n and shows how (with the exception of the Lie algebra of type B_n) these can be constructed, in their natural representations, using the combinatorics of heaps. Chapter 2 develops the theory of heaps over Dynkin diagrams. These heaps can be thought of (and drawn) as partially ordered sets whose elements are labelled by vertices of the Dynkin diagram, subject to certain rules. This combinatorial theory underpins the approach of the rest of the book. Chapter 3 explains how to associate algebraic objects with heaps, including faithful permutation representations of finite and affine Weyl groups. Chapter 4 develops and recalls the basic ideas of Lie theory as it applies to Lie algebras and Weyl groups. Sections 4.2 and 4.3 summarize key results and definitions that are needed in the sequel. These sections are not self-contained, and are primarily based on the books of Carter [11], Humphreys [36] and Kac [37].

Chapter 5 defines minuscule representations in terms of the combinatorics of heaps. Chapter 6 proves the classification of full heaps over affine Dynkin diagrams. These heaps give rise to representations of affine Kac–Moody algebras that can be thought of as infinite dimensional analogues of minuscule representations. Chapter 7 constructs Chevalley bases and structure constants for simple Lie algebras over $\mathbb{C}$ in terms of heaps. Chapter 8 examines some combinatorial properties of the Weyl group in its action on the weights of a minuscule representation, and explicitly describes the orbits of ordered pairs under this action. We also discuss some combinatorial properties of the weight polytope, that is, the convex hull of the weights of a minuscule representation.

The jewel in the crown of minuscule representations is the 56-dimensional representation in type E_7, and Chapters 9 and 10 are devoted to a study of it and its close relatives. Chapter 9 discusses the combinatorics of the 28 bitangents to a plane quartic curve, which can be identified with opposite pairs of weights in the 56-dimensional

Minuscule representations also have applications to areas more remote from representation theory, including special functions theory (see [25] and references therein), random walks [2, 44], and conformal field theory: the paper [22] considers a correspondence between minimal fluxes and minuscule weights of Lie algebras in types A, D and E. Another application to physics appears in the paper [71], which constructs a set of lattices by decorating the root lattices of various Lie algebras with their minuscule representations. The paper studies a family of Hamiltonians of fermions hopping on these lattices in the presence of a background gauge field; in this context, the Hamiltonians are themselves elements of the Lie algebras acting in their minuscule representations.

As its title suggests, the focus of this book is on combinatorial properties of minuscule representations. This primarily refers to combinatorial properties of the weights of the representations, especially the action of the Weyl group by orthogonal transformations on the set of weights. Most of the literature on minuscule representations says very little about this action, other than that it is transitive. However, the details of the action turn out to be fascinating. For example, we shall see in Chapter 8 that the failure of this action in general to be doubly transitive gives the Weyl group an interesting structure as a permutation group, which is intimately related to (a) branching rules of minuscule representations, (b) the combinatorics of the polytope formed by the convex hull of the weights and (c) various well-known families of graphs. An important special case is when the weights are closed under negation and the Weyl group acts as a rank 4 permutation group. In this case, the action of the Weyl group on pairs of opposite weights is doubly transitive, but the failure in general of the action to be triply transitive leads to the interesting combinatorial features of the 28 bitangents to a plane quartic curve and the 27 lines on a cubic surface; these features include examples of structures such as 2-graphs and generalized quadrangles, as well as the rich combinatorics of Steiner complexes and Schläfli double sixes.

A main object of interest in this book is a certain type of infinite labelled poset known as a full heap. Full heaps are defined from generalized Cartan matrices using only combinatorics, and they can be used to construct affine analogues of minuscule representations. These give rise to faithful permutation representations of affine Weyl groups (some of which are familiar from other contexts) as well as minuscule-like representations of various affine Kac–Moody algebras. These representations are closely related to the combinatorics of the associated root system; among other things, this allows a construction of Chevalley bases for Lie algebras using the combinatorics of heaps, as discussed in Chapter 7. The heap-theoretic approach also makes certain features of minuscule representations of Lie algebras easy to understand, such as the construction of certain invariant symplectic and orthogonal forms. We also remark on the invariant cubic forms in type E_6 and invariant quartic forms in type E_7.

The final chapter contains a survey of some important combinatorial properties related to minuscule representations, including minuscule elements, Gaussian posets, and the jeu de taquin approach to Schubert calculus.

This book is not, and is not intended to be, a general introduction to Lie algebras or Weyl groups of finite and affine type. There are already plenty of good books on

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Classical Lie algebras and Weyl groups

One of the main goals of the first half of this book is to construct Lie algebras and Weyl groups from certain labelled partially ordered sets known as heaps. The formal definition of heaps is in terms of categories, which will be defined in Chapter 2. The purpose of Chapter 1 is to show how particular examples of heaps can be used to give combinatorial constructions of algebraic objects.

In Section 1.1, we summarize the basic properties of Lie algebras. In Section 1.2, we define the Lie algebras of types A_n, B_n, C_n and D_n in terms of matrices; that is, the Lie algebras $\mathfrak{sl}(n+1, V)$, $\mathfrak{so}(2n+1, V)$, $\mathfrak{sp}(2n, V)$ and $\mathfrak{so}(2n, V)$, respectively. Except in type B_n, the Lie algebra representations given by these matrices will turn out to be "minuscule". Because of this, they may easily be constructed in terms of heaps (shown in Figures 1.1, 1.2 and 1.3) as we explain in Section 1.3. Finally, in Section 1.4, we define the classical Weyl groups of types A_n, B_n and D_n as abstract groups. We also explain how to construct the groups using the same three infinite families of heaps shown in Section 1.3.

1.1 Lie algebras

A *Lie algebra* is a vector space $\mathfrak{g}$ over a field k equipped with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (the *Lie bracket*) satisfying the conditions

$$[x, x] = 0,$$
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0,$$

for all $x, y, z \in \mathfrak{g}$. (These conditions are known respectively as *antisymmetry* and the *Jacobi identity*.)

Example 1.1.1 Let A be any associative algebra over k. Then A can be made into a Lie algebra by replacing the associative multiplication $\circ$ by the Lie bracket

$$[x, y] := x \circ y - y \circ x.$$

Exercise 1.1.2 Verify that the Lie bracket defined in Example 1.1.1 does indeed satisfy the antisymmetry property and the Jacobi identity.

Example 1.1.3 One of the most important examples of a Lie algebra is $\mathfrak{gl}_n(k)$. This is the Lie algebra obtained from the associative matrix algebra $M_n(k)$ of all $n \times n$

matrices with entries in k by applying the construction of Example 1.1.1; in other words, the elements of $\mathfrak{gl}_n(k)$ are all $n \times n$ matrices over k and the Lie bracket $[A, B]$ is defined by $[A, B] = AB - BA$.

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras over a field k, then a *homomorphism* of Lie algebras from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ is a k-linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}_1$. An *isomorphism* of Lie algebras is a bijective homomorphism.

If V is any vector space over k then the Lie algebra $\mathfrak{gl}(V)$ is the k-vector space of all k-linear maps $T : V \to V$, equipped with the Lie bracket satisfying

$$[T, U] := T \circ U - U \circ T,$$

where $\circ$ is composition of maps.

A *representation* of a Lie algebra $\mathfrak{g}$ over k is a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ for some k-vector space V. In this case, we call V a (left) *module* for the Lie algebra $\mathfrak{g}$ (or a $\mathfrak{g}$-module, for short) and we say that V *affords* ρ. If $x \in \mathfrak{g}$ and $v \in V$, we write $x.v$ to mean $\rho(x)(v)$. The *dimension* of a module (or of the corresponding representation) is the dimension of V. If ρ is the zero map, then the representation ρ and the module V are said to be *trivial*.

A *submodule* of a $\mathfrak{g}$-module V is a k-subspace W of V such that $x.w \in W$ for all $x \in \mathfrak{g}$ and $w \in W$. If V has no submodules other than itself and the zero submodule, then V is said to be *irreducible*. If W is a submodule of V, the quotient vector space V/W acquires a well-defined $\mathfrak{g}$-module structure via the condition $x.(v + W) = (x.v) + W$; this is known as a *quotient module*.

If V_1 and V_2 are $\mathfrak{g}$-modules, then a k-linear map $f : V_1 \to V_2$ is called a *homomorphism of* $\mathfrak{g}$*-modules* if $f(x.v) = x.f(v)$ for all $x \in \mathfrak{g}$ and $v \in V_1$. An *isomorphism of* $\mathfrak{g}$*-modules* is an invertible homomorphism of $\mathfrak{g}$-modules.

A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a *subalgebra* of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. If, furthermore, we have $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$ (or, equivalently, $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$) then $\mathfrak{h}$ is said to be an *ideal* of $\mathfrak{g}$. We write $\mathfrak{h} \leq \mathfrak{g}$ (respectively, $\mathfrak{h} \trianglelefteq \mathfrak{g}$) to mean that $\mathfrak{h}$ is a subalgebra (respectively, an ideal) of $\mathfrak{g}$. If S is a subset of $\mathfrak{g}$, then the smallest subalgebra of $\mathfrak{g}$ containing S is called the *subalgebra generated by* S, and the elements of S are called *generators* of the subalgebra. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{h}$ becomes a $\mathfrak{g}$-module via the action $g.h = [g, h]$. The quotient module $\mathfrak{g}/\mathfrak{h}$ then inherits a well-defined Lie algebra structure via the condition $[x + \mathfrak{h}, y + \mathfrak{h}] = [x, y] + \mathfrak{h}$; we call such a Lie algebra a *quotient Lie algebra* of $\mathfrak{g}$.

If $\mathfrak{g}$ has no ideals other than itself and the zero ideal, then $\mathfrak{g}$ is said to be *simple*. A simple Lie algebra of dimension 1 is called a *trivial simple Lie algebra*. The *derived algebra*, $\mathfrak{g}'$, of a Lie algebra $\mathfrak{g}$ is the subalgebra generated by all elements $\{[x_1, x_2] : x_1, x_2 \in \mathfrak{g}\}$. It can be shown that $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$.

Example 1.1.4 Let V be an n-dimensional k-vector space. Since we can identify $M_n(k)$ with $\mathrm{End}(V)$, we can also identify $\mathfrak{gl}_n(k)$ with $\mathfrak{gl}(V)$; note that both identifications rely on the choice of a basis for V. This identification, which is an isomorphism of Lie algebras $\rho : \mathfrak{gl}_n(k) \to \mathfrak{gl}(V)$, endows V with the structure of a $\mathfrak{gl}_n(k)$-module. More precisely, we have

$$\rho([A, B]) = \rho(A) \circ \rho(B) - \rho(B) \circ \rho(A),$$

where the map $\circ$ on the right hand side refers to composition in the associative algebra $\mathrm{End}(V)$. The $\mathfrak{gl}_n(k)$-module structure on V satisfies $A.v = \rho(A)(v)$, and can be identified with left multiplication of the vector v by the matrix A, once a basis of V has been chosen. The dimension of the module V (or the representation ρ) is n.

Recall that the *trace* of a matrix is the sum of its diagonal entries. A standard property of matrices (see Exercise 1.1.5 below) is that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ when A and B are square matrices of the same size. It follows that conjugate matrices have the same trace, and thus that we may speak of the trace of an endomorphism of a vector space without reference to a basis. Using the formula for $\rho([A, B])$ in the previous paragraph, we see that

$$\mathrm{tr}\rho([A, B]) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0.$$

It follows from this that if we define $\mathfrak{sl}_n(k)$ to be the subspace of $\mathfrak{gl}_n(k)$ consisting of all matrices A with $\mathrm{tr}(A) = 0$, then $\mathfrak{sl}_n(k) \trianglelefteq \mathfrak{gl}_n(k)$. (It is not too hard to show that $\mathfrak{sl}_n(k)$ is equal to the derived algebra of $\mathfrak{gl}_n(k)$, by considering a suitable basis of $\mathfrak{sl}_n(k)$.) Since $\mathfrak{sl}_n(k)$ has dimension $n^2 - 1$ and $\mathfrak{gl}_n(k)$ has dimension n^2, the quotient module $\mathfrak{gl}_n(k)/\mathfrak{sl}_n(k)$ is one-dimensional, and has the trivial module structure. In the case where $k = \mathbb{C}$, it can be shown that $\mathfrak{sl}_n(\mathbb{C})$ is a simple Lie algebra. The identification of $\mathfrak{gl}_n(k)$ with $\mathfrak{gl}(V)$ of Example 1.1.4 identifies the subspace $\mathfrak{sl}_n$ with a subspace of $\mathfrak{gl}(V)$, which we will call $\mathfrak{sl}(n, V)$.

Exercise 1.1.5 Prove that if A and B are two $n \times n$ matrices then we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Deduce that if P is invertible, then $\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(A)$.

We can make the Lie algebra $\mathfrak{g}$ into a module over itself, in which $x.y := [x, y]$. This is known as the *adjoint module*, and the corresponding representation is called the *adjoint representation*. If $x \in \mathfrak{g}$, we define the linear map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}_x(y) := [x, y]$. (We may also write $(\mathrm{ad}\, x)(y)$ for $\mathrm{ad}_x(y)$ to avoid excessive subscripts.)

If $\mathfrak{g}$ is a Lie algebra over k, then a *derivation* of $\mathfrak{g}$ is a k-linear map $D : \mathfrak{g} \to \mathfrak{g}$ satisfying Leibniz's law, namely

$$D([x, y]) = [D(x), y] + [x, D(y)].$$

The Jacobi identity guarantees that for each $x \in \mathfrak{g}$, the map $D_x : \mathfrak{g} \to \mathfrak{g}$ given by

$$D_x(y) = \mathrm{ad}_x(y) = [x, y]$$

is a derivation.

If B is a k-basis for the Lie algebra $\mathfrak{g}$, then for any $b_i, b_j \in B$, we may write

$$[b_i, b_j] = \sum_{i \in B} \lambda_{ij}^k b_k,$$

where we have $\lambda_{ij}^k \in k$. The scalars λ_{ij}^k are known as the *structure constants* of $\mathfrak{g}$ with respect to B. (The concept of structure constants applies to any k-algebra with a distinguished basis.)

Exercise 1.1.6 Suppose that $\mathfrak{g}$ is a simple Lie algebra. Show that we have $\mathfrak{g}' = 0$ if $\mathfrak{g}$ is trivial, and $\mathfrak{g}' = \mathfrak{g}$ otherwise.

Exercise 1.1.7 Verify that the adjoint module for $\mathfrak{g}$ is indeed a module.

Exercise 1.1.8 Show that if $\mathfrak{g}$ is a Lie algebra and $x, y \in \mathfrak{g}$, then

$$\mathrm{ad}_{[x,y]} = (\mathrm{ad}_x \circ \mathrm{ad}_y) - (\mathrm{ad}_y \circ \mathrm{ad}_x).$$

Exercise 1.1.9 Show that if $V_1, V_2, \dots, V_n$ are $\mathfrak{g}$-modules over the field k, then the tensor product $V_1 \otimes_k V_2 \otimes_k \cdots \otimes_k V_n$ becomes a $\mathfrak{g}$-module under the action

$$x.(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} (v_1 \otimes \cdots \otimes v_{i-1} \otimes x.v_i \otimes v_{i+1} \otimes \cdots \otimes v_n).$$

1.2 The classical Lie algebras

Some other important examples of Lie algebras can be defined as subalgebras of $\mathfrak{gl}(V)$ that respect certain invariant bilinear forms. In this section, we will explain exactly what this means.

Definition 1.2.1 Let $\mathfrak{g}$ be a Lie algebra over a field k and let V be a $\mathfrak{g}$-module. A bilinear map $B : V \otimes_k V \to k$ is said to be $\mathfrak{g}$-*invariant* if for all $x \in \mathfrak{g}$ and $v \in V$, we have

$$B(x.v, v) + B(v, x.v) = 0.$$

The *radical*, $\mathrm{rad}(B)$ of B is defined to be the subset of V given by

$$\{r \in V : B(r, v) = 0 \text{ for all } v \in V\}.$$

If the $\mathrm{rad}(B)$ is zero, we call B *nondegenerate*. If k does not have characteristic 2, a nondegenerate bilinear form B is said to be *orthogonal* (respectively, *symplectic*) if for all $v_1, v_2 \in V$ we have $B(v_1, v_2) = \epsilon B(v_2, v_1)$, where $\epsilon = 1$ (respectively, $\epsilon = -1$).

Exercise 1.2.2 Show that if the elements x and y satisfy the invariance conditions of Definition 1.2.1, then so does $[x, y]$.

Exercise 1.2.3 Let B be a bilinear form on an $\mathfrak{g}$-module V. Show that $\mathrm{rad}(B)$ is a $\mathfrak{g}$-submodule of V. Deduce that if V is a simple $\mathfrak{g}$-module and B is not the zero map, then B is nondegenerate.

If J is an $n \times n$ matrix over the field k, we may use J to define a bilinear form by

$$(x, y) := x^{\mathrm{T}} J y,$$

where T denotes transpose.

If V is an n-dimensional $\mathfrak{g}$-module with associated representation ρ, the condition for the aforementioned bilinear form to be $\mathfrak{g}$-invariant is to have

$$(g.x, y) + (x, g.y) = 0$$

for all g, x and y. This is equivalent to the condition

$$x^{\mathrm{T}} \rho(g)^{\mathrm{T}} J y + x^{\mathrm{T}} J \rho(g) y = 0$$

for all x and y; in other words, $\rho(g)^{\mathrm{T}} J + J\rho(g) = 0$ for all matrices $\rho(g)$.

Proposition 1.2.4 *Let V be an n-dimensional vector space over k and let J be an $n \times n$ matrix over k. Then the elements $g \in \mathfrak{gl}(V)$ that satisfy*

$$(g.x, y) + (x, g.y) = 0$$

for all $x, y \in V$ form a Lie subalgebra of $\mathfrak{gl}(V)$.

Proof Let g and h be elements of $\mathfrak{gl}(V)$ that satisfy the hypotheses. We need to show that $[g, h]$ has the same property; in other words, that

$$(g.(h.x), y) - (h.(g.x), y) + (x, g.(h.y)) - (x, h.(g.y)) = 0.$$

It follows from the hypotheses that

$$(h.x, y) + (x, h.y) = 0,$$

which in turn implies that

$$(g.(h.x), y) + (h.x, g.y) + (g.x, h.y) + (x, g.(h.y)) = 0.$$

Reversing the roles of g and h gives

$$(h.(g.x), y) + (g.x, h.y) + (h.x, g.y) + (x, h.(g.y)) = 0.$$

Subtracting these last two equations from each other completes the proof. □

Definition 1.2.5 Let V be an n-dimensional vector space over k and let J be an $n \times n$ matrix over k. We define the Lie subalgebra $\mathfrak{gl}_J(V)$ of $\mathfrak{gl}(V)$ to be the set of elements $g \in \mathfrak{gl}(V)$ that satisfy

$$(g.x, y) + (x, g.y) = 0$$

for all $x, y \in V$.

Example 1.2.6 Let I_n be the $n \times n$ identity matrix, and let J_C be the $2n \times 2n$ block matrix of the form

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Let V be a $2n$-dimensional vector space over k, and let g be a typical element of $\mathfrak{gl}(V)$. As a block matrix, we may write

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

As mentioned previously, a necessary and sufficient condition for $g \in \mathfrak{gl}_{J_C}(V)$ is for

$$g^T J_C + J_C g = 0,$$

that is,

$$\begin{pmatrix} -C^T & A^T \\ -D^T & B^T \end{pmatrix} + \begin{pmatrix} C & D \\ -A & -B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that B and C are symmetric $n \times n$ matrices, and that $A = -D^T$. Since the space of symmetric $n \times n$ matrices has dimension $n(n+1)/2$, it follows that the dimension of $\mathfrak{gl}_{J_C}(V)$ is given by

$$2.\frac{n(n+1)}{2} + n^2 = n(2n+1).$$

If we equip V with the obvious ordered basis

$$(e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n),$$

then the bilinear form B_C associated with J_C has the following properties:

$$\begin{aligned}(e_i, e_j) &= 0 \text{ for all } i, j,\\ (f_i, f_j) &= 0 \text{ for all } i, j,\\ (e_i, f_j) &= \delta_{i,j},\\ (f_i, e_j) &= -\delta_{i,j},\end{aligned}$$

where δ is the Kronecker delta, meaning that $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise. The fact that J_C is invertible means that this bilinear form is nondegenerate. It follows by bilinearity that the form B_C is alternating; in other words, we have $(x, x) = 0$ for all $x \in V$. Expanding $(x + y, x + y)$ then shows that the form is *skew-symmetric*; in other words, we have $(x, y) = -(y, x)$ for all $x, y \in V$. It follows that B_C is symplectic.

Definition 1.2.7 The Lie algebra $\mathfrak{gl}_{J_C}(V)$ of Example 1.2.6 is known as the *symplectic Lie algebra*, and is denoted by $\mathfrak{sp}(2n, V)$.

Example 1.2.8 Let I_n be the $n \times n$ identity matrix, and let J_D be the $2n \times 2n$ block matrix of the form

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Let V be a $2n$-dimensional vector space over k, and let g be a typical element of $\mathfrak{gl}(V)$. As in Example 1.2.6, we may write

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The necessary and sufficient condition for $g \in \mathfrak{gl}_{J_D}(V)$, namely

$$g^T J_D + J_D g = 0,$$

now becomes

$$\begin{pmatrix} C^T & A^T \\ D^T & B^T \end{pmatrix} + \begin{pmatrix} C & D \\ A & B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that B and C are skew-symmetric $n \times n$ matrices, and that $A = -D^T$. Since the space of skew-symmetric $n \times n$ matrices has dimension $n(n-1)/2$, it follows that the dimension of $\mathfrak{gl}_{J_D}(V)$ is given by

$$2.\frac{n(n-1)}{2} + n^2 = n(2n-1).$$

If we equip V with the obvious ordered basis

$$(e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n),$$