Part 1

Transmission lines using a distributed equivalent circuit

1 Pulses on transmission lines

The term 'transmission line' is not uniquely defined and is usually taken to mean a length of line joining a source to a termination. A simple example of such a transmission line is shown in Figure 1.1.

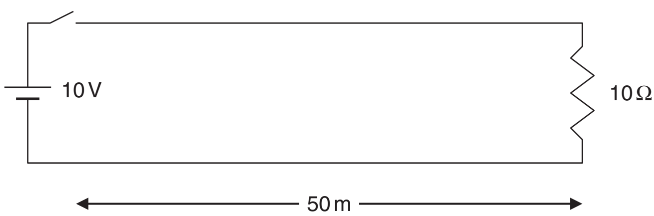

Figure 1.1 A simple transmission line 50 m long joining a 10 V source to a 10 Ω termination.

This transmission line might consist of just two parallel wires or a coaxial cable or something more unusual. Using ordinary equivalent circuit theory, which has widespread use in both Electrical and Electronic Engineering, a current of 1 A flows in the whole circuit, the moment the switch is closed. This is only true if every element in the circuit is considered to be a lumped element. Now a lumped element is a circuit component in which a current is instantaneously produced as soon as a voltage is applied. The battery and the resistor may well have a small delay before any current appears, but in this introductory chapter it will be assumed these effects can be neglected. However, the 50 m length of line cannot be described as a lumped element as it takes a finite time for a voltage introduced at one end to propagate to the other. Only if this time is much smaller than any other transient being considered can the transmission line effects described in this chapter be neglected. In order to analyse this propagation, a distributed circuit is needed which gives some considerable insight into the performance of transmission lines. This circuit approach is limited to the use of voltages and currents, which are inappropriate for transmission lines like waveguides and optical fibres. In the later chapters, two alternative descriptions of the lines will be given; one in terms of electromagnetic fields and the other in terms of photons. All three descriptions reveal unique aspects of the propagation along transmission lines and together they give a more complete picture.

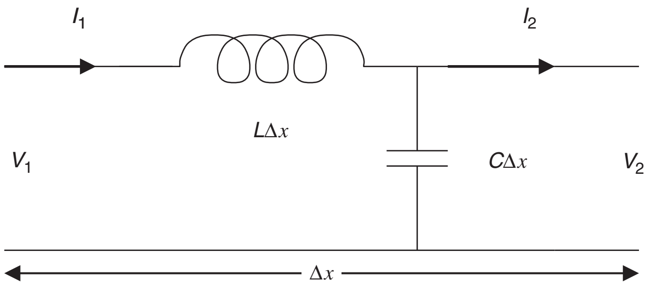

Figure 1.2 An equivalent circuit of a small length, Δx, of loss-less transmission line.

The distributed circuit method applies to those transmission lines which have two or more conductors. However, this chapter will discuss the simplest case of just two conductors, which is the most common configuration. There are two assumptions that are needed before a distributed circuit can be developed. The first is that the conductors must be good conductors so that the voltage does not vary around the cross-section of an individual line, or in other words, it is an equipotential surface. Secondly, the lines will be assumed to be free from losses so that signals can travel down them without attenuation. In Chapter 6, the effects of various types of losses on the operation of the lines will be introduced. Electrically, there are only two things left to consider; they are the capacitance and the inductance of the line. Since a transmission line normally is assumed to be uniform, both of these quantities increase with length, so it is common practice to define them for a one-metre length of line. They are then called the distributed capacitance, C, and distributed inductance, L, and have the units of Fm^{-1} and Hm^{-1} respectively.

In practice, these parameters can easily be measured using a one-metre length of the line. If the far end is open-circuited, then the capacitance, C, can be measured at the near end, with some corrections for the capacitance of the open circuit. Similarly, if the far end is short-circuited, the inductance, L, can also be measured, again with an appropriate correction for the inductance of the short circuit. These measurements can be made over a wide range of frequencies and, to get started on this basic description, these distributed parameters will be assumed to be constant with frequency. In Chapter 6 the variation of these parameters with frequency, called dispersion, will be discussed. So an equivalent circuit for a short length of line, Δx, is fairly simple and is shown in Figure 1.2.

The short length of line is important because, as $\Delta x \to 0$, the two distributed elements become effectively lumped elements again and so ordinary circuit equations can legitimately be applied. This sounds like a circular argument, but the principle being applied is to do with the transit time of signals. If a length of line is small enough, this transit time can be neglected and so ordinary circuit theory can be used. In the electromagnetic field description, given in later chapters, this somewhat circuitous argument is not required!

1.1 Velocity and characteristic impedance

Imagine then a voltage source, V_1, is connected to the left-hand side of a short length of transmission line, as shown in Figure 1.2, which also causes a current, I_1, to flow as shown. As the signal travels down the short length, two changes occur. Firstly, there is a voltage drop across the inductance and secondly, a current loss through the capacitor. So at the right-hand side the voltage and current become:

$$V_2 = V_1 - L\Delta x \frac{\partial I}{\partial t} \text{ and } I_2 = I_1 - C\Delta x \frac{\partial V}{\partial t}, \tag{1.1}$$

where these are obtained from the usual circuit laws for capacitances and inductances. These equations would be the same for an equivalent circuit with the inductance on the right-hand side of the capacitance.

The equations can be rearranged as follows:

$$V_2 - V_1 = \Delta V = -L\Delta x \frac{\partial I}{\partial t} \text{ and } I_2 - I_1 = \Delta I = -C\Delta x \frac{\partial V}{\partial t},$$

$$\frac{\Delta V}{\Delta x} = -L\frac{\partial I}{\partial t} \text{ and } \frac{\Delta I}{\Delta x} = -C\frac{\partial V}{\partial t}, \tag{1.2}$$

and in the limit as $\Delta x \to 0$:

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \text{ and } \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}. \tag{1.3}$$

These are called the Telegraphists' equations and are useful in linking the voltage and current on a transmission line. However, since they are cross-linked in these two variables, it is not possible directly to eliminate one or other of them to find a solution. The normal route to a solution is to differentiate each of them with respect to both time and distance:

$$\frac{\partial^2 V}{\partial x^2} = -L\frac{\partial^2 I}{\partial x \partial t} \text{ and } \frac{\partial^2 V}{\partial t \partial x} = -L\frac{\partial^2 I}{\partial t^2},$$

$$\frac{\partial^2 I}{\partial x^2} = -C\frac{\partial^2 V}{\partial x \partial t} \text{ and } \frac{\partial^2 I}{\partial t \partial x} = -C\frac{\partial^2 V}{\partial t^2}. \tag{1.4}$$

Since the parameters of space and time are independent, the order of the differentiation is not important. So eliminating the mixed differentials gives:

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \text{ and } \frac{\partial^2 I}{\partial x^2} = CL\frac{\partial^2 I}{\partial t^2}. \tag{1.5}$$

These equations are called wave equations because their solutions are waves. Both the voltage and the current obey the same equation in this simple case. The solutions of these wave equations are any functions of the variable:

$$t \pm \frac{x}{v}, \text{i.e. } V = f\left(t \pm \frac{x}{v}\right) \text{ and } I = g\left(t \pm \frac{x}{v}\right), \tag{1.6}$$

where v is a constant.

Substituting the voltage function into the wave equations gives

$$\frac{1}{v^2}f''\left(t \pm \frac{x}{v}\right) = LCf''\left(t \pm \frac{x}{v}\right).$$

So the constant, v, is given by:

$$v = \frac{1}{\sqrt{LC}}. \tag{1.7}$$

By examining Equations (1.6) and taking the minus sign it can be seen that any function of t is delayed in the positive x direction. This function is called a forward wave and its velocity is given by v in Equation (1.7). The positive sign is for waves moving in the negative x direction and these are called backward or reflected waves. Now the link between the voltage and the current waves is found by using the Telegraphists' equations in (1.3). Substituting the functions given in Equation (1.6) gives

$$\pm\frac{1}{v}f'\left(t \pm \frac{x}{v}\right) = -Lg'\left(t \pm \frac{x}{v}\right) \text{ and } \pm\frac{1}{v}g'\left(t \pm \frac{x}{v}\right) = -Cf'\left(t \pm \frac{x}{v}\right).$$

Integrating both sides of the equations with respect to time gives

$$\pm\frac{1}{v}f\left(t \pm \frac{x}{v}\right) = -Lg\left(t \pm \frac{x}{v}\right) \text{ and } \pm\frac{1}{v}g\left(t \pm \frac{x}{v}\right) = -Cf\left(t \pm \frac{x}{v}\right). \tag{1.8}$$

Then using Equations (1.6) and (1.7) gives

$$\frac{V}{I} = \pm\sqrt{\frac{L}{C}}$$

from both equations in (1.8).

The positive sign relates to the forward waves and the negative sign to the reverse waves. This ratio of voltage to current has the units of ohms in this case and is normally given the symbol Z_0 and called the characteristic impedance of the transmission line. By denoting a subscript *plus* to forward waves and a subscript *minus* to reverse waves gives

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{V_+}{I_+} = -\frac{V_-}{I_-}. \tag{1.9}$$

The negative sign in Equation (1.9) is because the wave is travelling in the reverse direction.

1.2 Reflection coefficient

The next concept to consider is the reflection of waves. This is often caused by a sudden change of impedance along a transmission line. The simplest case is a line terminated with an impedance, Z_L, which will cause reflections because the total voltage across the impedance, V_L, and the current through it, I_L, is given by Ohm's Law as

$$\frac{V_L}{I_L} = Z_L = \frac{V_+ + V_-}{I_+ + I_-}. \tag{1.10}$$

This assumes that Z_L has small dimensions so that there is zero transit time between the arrival of the wave and the appearance of a current in the impedance. In other words it is subject to the normal circuit laws. The presence of the reflected wave enables Ohm's Law to be obeyed both at the termination and in the two waves given in Equation (1.9). In the time domain, a termination may not always be described as a simple impedance, Z_L, whereas in the frequency domain it is always possible. In many of the examples that follow, Z_L is a pure resistance, and so Equation (1.10) is valid for both domains. However, for the later examples, 1.9 onwards, more complex time domain expressions are developed.

A special case for Equation (1.10) is when:

$$Z_L = Z_0. \tag{1.11}$$

Then Z_L is called a matched termination and, since it is equal to the characteristic impedance, no reflections occur. A useful measure of the amount of reflection is the ratio, ρ, of the reflected to the incident voltage wave:

$$\rho = \frac{V_-}{V_+} = -\frac{I_-}{I_+}. \tag{1.12}$$

In the third part of Equation (1.12) the negative sign is because of the negative sign in Equation (1.9). So the current reflects in an equal and opposite way to the voltage. Substituting Equation (1.12) into Equation (1.10) gives

$$Z_L = \frac{V_+(1+\rho)}{\frac{V_+}{Z_0}(1-\rho)} \quad \text{or} \quad \rho = \frac{Z_L - Z_0}{Z_L + Z_0}. \tag{1.13}$$

It is useful to realise the significance of ρ or, as it is commonly called, the reflection coefficient. In the frequency domain, both Z_L and Z_0 can be complex, making the reflection coefficient complex as well.

If we limit the discussion to real values of Z_0 and values of Z_L where the real part is positive, then

$$|\rho| \leq 1 \quad \text{and} \quad -\pi \leq \angle\rho \leq \pi. \tag{1.14}$$

For example, some typical values are:

Z_L/Z_0	$\lvert\rho\rvert$	$\angle\rho$
0 (short circuit)	1	$\pm\pi$
∞ (open circuit)	1	0
1 (matched load)	0	indeterminate
ja (inductance)	1	π to 0
−jb (capacitance)	1	$-\pi$ to 0
2 (resistance $>Z_0$)	$\frac{1}{3}$	0
0.5 (resistance $<Z_0$)	$\frac{1}{3}$	$\pm\pi$

where a and b are constants.

Only when there is a resistive part to Z_L does the amplitude of ρ go below unity. When Z_L is reactive the phase or argument of ρ can be as large as π, due to the bilinear nature of Equation (1.13). So far in the discussion, the nature of the voltage waveform has been left totally general. For instance, it could be a voltage step, which would occur when a battery is connected, or it could be a sine wave, or a pulse or some rarer wave like a bi-pulse. To illustrate these waveforms in the time domain, a series of problems will now be briefly described, starting with only resistive terminations and ending with more complex terminations. The next chapter will discuss problems involving sinusoidal waves.

1.3 Step waves incident on resistive terminations

Example 1.1 A powerful ten-volt battery is suddenly connected to a 100 m long transmission line. At the far end of the line is a short circuit. If the velocity of propagation is 2.10^8 ms^{-1} and the characteristic impedance is 50 Ω, find the current in the short circuit after 5 μs, assuming the battery has zero internal resistance. See Figure 1.3.

Solution to Example 1.1

This problem would have a simple solution in circuit theory, as the current would instantaneously be infinite! However, in transmission line theory, this is not the correct solution. Initially the battery sends a voltage step of amplitude 10 V and

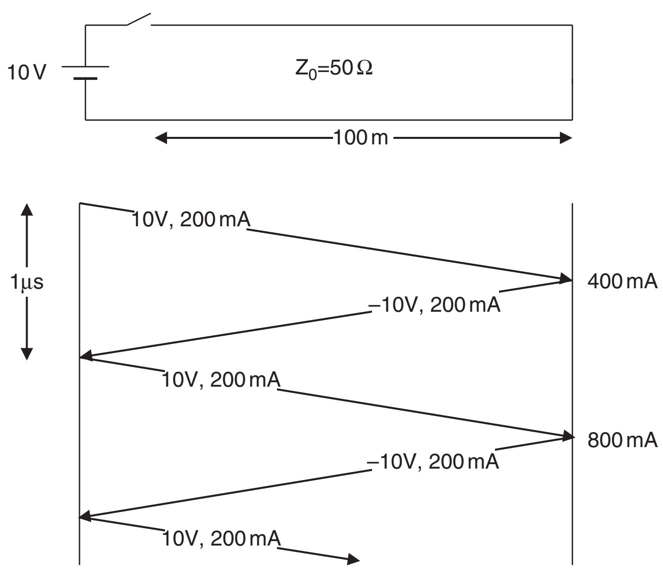

Figure 1.3 The circuit diagram and wave diagram for Example 1.1.

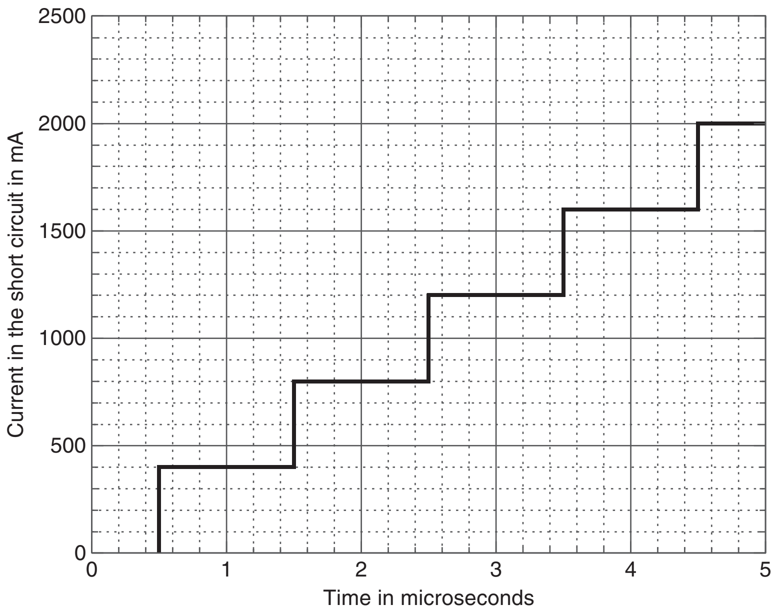

Figure 1.4 The current in the short circuit against time for Example 1.1.

current 200 mA (see Equation (1.9)) which takes 0.5 μs to reach the short circuit. Using Equations (1.12) and (1.13), this step is reflected so that reverse wave has a voltage of –10 V and a current of +200 mA. Thus the current in the short circuit jumps up to a total of 400 mA but with no overall voltage as the two waves superimpose so as to cancel their voltages and add their currents. The reflected wave returns after a similar reflection at the battery in 1.5 μs to add a further 400 mA to the current in the short circuit. This is shown in the wave diagram in Figure 1.3 where time is the coordinate vertically downwards. So just after 4.5 μs the current will be 2 A, as shown in Figure 1.4. Obviously the current will be limited by several factors, for instance, if the maximum current that the battery could supply was 40 A, then this current would be reached in 100 μs.

The essential thing to notice from this problem is that the transmission line limits the initial supply of current from the source and also that the step wave goes back and forth, endlessly delivering increases in current until a limit is reached. The battery supplies the original 200 mA continuously and the current builds up because of the wave motion, which keeps increasing the battery current in steps. So a ‘long’ short circuit could prevent dangerous currents for a short period.

It is useful to note that the battery is effectively ‘seeing’ a changing resistive load which reduces in value with time, as shown in Figure 1.5. If the line had been very long the battery would have just supplied 200 mA to the line. However, the multiple reflections in this example result in an increasing current being drawn from the battery.

Finally, the wave induces a positive current in one of the lines and a negative current in the other line. Thus the battery is sending out a current from one of its

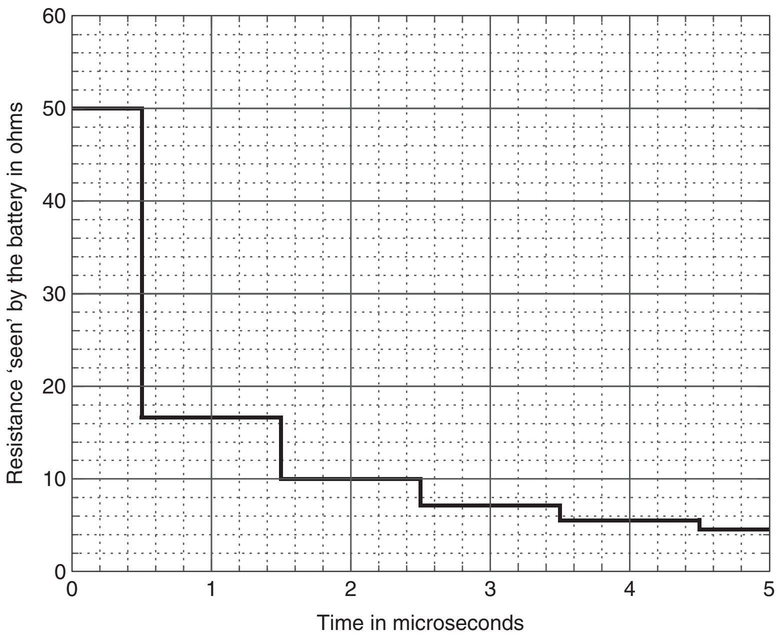

Figure 1.5 The resistance 'seen' by the battery against time in Example 1.1.

terminals and receiving a current at the other, as the wave progresses. It is usually easier to think of the upper wire in Figure 1.3 carrying the positive current of 200 mA and the lower or return wire a negative current of 200 mA.

Example 1.2 Using the same circuit as in Example 1.1, the switch is closed at $t = 0$ as before, but then opened at $t = 5\,\mu s$. Find the current waveform in the short circuit for the next $5\,\mu s$.

Solution to Example 1.2

This problem again would have a simple solution in circuit theory: the current would be zero. However, the wave theory does not give that answer. When the switch is opened, the wave is trapped in the circuit and cannot escape. When it reflects from the switch, which is now effectively an open circuit, the total wave goes on reflecting back and forth forever. In practice there will be some loss mechanisms, which will reduce the wave amplitude eventually to zero, but in this special case with no losses the circuit becomes a square wave oscillator. The solution is easier to see if all the five waves are added together at the moment the switch is opened. The total wave approaching the short circuit has an amplitude of 50 V and carries a current of 1 A. The total wave departing from the short circuit has an amplitude of –50 V and also carries a current of 1 A. These two