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A brief review of quantum mechanics

> Come forth into the light of things,
> Let nature be your teacher.
> –*William Wordsworth,*
> The Tables Turned

The main focus of this book is many-particle systems such as electrons in a crystal. Such systems are studied within the framework of quantum mechanics, with which the reader is assumed to be familiar. Nevertheless, a brief review of this subject will provide an opportunity to establish notation and collect results that will be used later on.

1.1 The postulates

Quantum mechanics is based on five postulates, listed below with some explanatory comments.

(I) The quantum state

The quantum state of a particle, at time t, is described by a continuous, single-valued, square-integrable wave function $\Psi(\mathbf{r}, t)$, where $\mathbf{r}$ is the position of the particle. In Dirac notation, the state is represented by a state vector, or ket, $|\Psi(t)\rangle$, which is an element of a vector space V. We define a dual vector space V* whose elements, called bras, are in one-to-one correspondence with the elements of V: ket $|\alpha\rangle \in \mathrm{V} \leftrightarrow$ bra $\langle\alpha| \in \mathrm{V}^*$, as illustrated in Figure 1.1. The bra corresponding to ket $c\,|\alpha\rangle$ is $c^*\langle\alpha|$, where c^* is the complex conjugate of c. The inner product of kets $|\alpha\rangle$ and $|\beta\rangle$ is denoted by $\langle\beta|\alpha\rangle$, and it is a complex number (c-number). Note that the inner product is obtained by combining a bra and a ket. By definition, $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$. The state vectors $|\Psi(t)\rangle$ and $c\,|\Psi(t)\rangle$, where c is any nonzero complex number ($c \in \mathbb{C} - \{0\}$), describe the same physical state; because of that, the state

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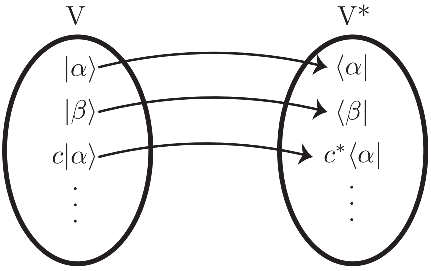

Figure 1.1 Vector space V of kets and the corresponding dual space V* of bras. A one-to-one correspondence exists between kets and bras.

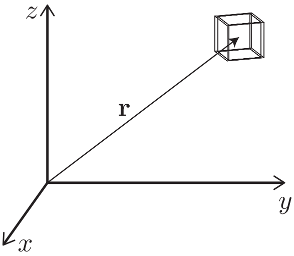

Figure 1.2 The probability of finding the particle, at time t, in the cube of volume d^3r, centered on $\mathbf{r}$, is $|\Psi(\mathbf{r}, t)|^2 d^3r$.

is usually taken to be normalized to unity: $\langle\Psi(t)|\Psi(t)\rangle = 1$. The normalized wave function has a probabilistic interpretation: $\Psi(\mathbf{r}, t)$ is the probability amplitude of finding the particle at position $\mathbf{r}$ at time t; this means that $|\Psi(\mathbf{r}, t)|^2 d^3r$ is the probability of finding the particle, at time t, in the infinitesimal volume d^3r centered on point $\mathbf{r}$ (see Figure 1.2).

Note that the description of a quantum state is completely different from the one used in classical mechanics, where the state of a particle is specified by its position $\mathbf{r}$ and momentum $\mathbf{p}$ at time t.

(II) Observables

An observable is represented by a linear, Hermitian operator acting on the state space. If A is an operator, being linear means that

$$A\left(c_1|\alpha\rangle + c_2|\beta\rangle\right) = c_1 A|\alpha\rangle + c_2 A|\beta\rangle, \quad |\alpha\rangle, |\beta\rangle \in \mathrm{V}, \quad c_1, c_2 \in \mathbb{C},$$

and being hermitian means that $A^\dagger = A$, where $A^\dagger$ is the adjoint of A, defined by the relation

$$\langle\beta|A^\dagger|\alpha\rangle = \langle\alpha|A|\beta\rangle^*.$$

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In particular, the position of a particle is represented by the operator **r**, its momentum **p** by $-i\hbar\nabla$, and its energy by the Hamiltonian operator H,

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t). \tag{1.1}$$

$V(\mathbf{r}, t)$ is the operator that represents the potential energy of the particle, m is the particle's mass, and $\hbar$ is Planck's constant h divided by 2π.

As with states, the representation of observables in quantum mechanics is completely different from that of their classical counterparts, which are simply represented by their numerical values.

(III) Time evolution

The state $|\Psi(t)\rangle$ of a system evolves in time according to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle. \tag{1.2}$$

If the Hamiltonian H does not depend explicitly on time, then

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle. \tag{1.3}$$

The operator $e^{-iHt/\hbar}$ is called the time evolution operator. Defining the stationary states $|\phi_n\rangle$ as the solutions of the eigenvalue equation, known as the time-independent Schrödinger equation,

$$H|\phi_n\rangle = E_n|\phi_n\rangle, \tag{1.4}$$

it is readily verified that $|\phi_n\rangle e^{-iE_nt/\hbar}$ is a solution of Eq. (1.2); the general solution of Eq. (1.2), when H is independent of t, is then given by

$$|\Psi(t)\rangle = \sum_n c_n|\phi_n\rangle e^{-iE_nt/\hbar}.$$

In contrast, the evolution of the classical state of a particle is determined by Hamilton's function H via Hamilton's equations of motion which, in one dimension, are

$$\dot{x} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial x. \tag{1.5}$$

(IV) Measurements

Let an observable be represented by the linear, Hermitian operator A, and consider the eigenvalue equation

$$A|\phi_n\rangle = a_n|\phi_n\rangle, \tag{1.6}$$

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where $a_1, a_2, \ldots$ are the eigenvalues, and $|\phi_1\rangle, |\phi_2\rangle, \ldots$ the corresponding eigenvectors, or eigenkets. In general, there may be infinitely many eigenvalues and eigenkets. If k eigenkets correspond to the same eigenvalue a_l, then a_l is said to be k-fold degenerate. The following is postulated:

1. The outcome of any measurement of A is always one of its eigenvalues.
2. The eigenkets $|\phi_1\rangle, |\phi_2\rangle, \ldots$ form a complete set of states, i.e., they form a basis set that spans the state vector space.
3. If the state of a system is described by the normalized state vector $|\Psi(t)\rangle$, and if the states $|\phi_1\rangle, |\phi_2\rangle, \ldots$ are orthonormal, then the probability of finding the system in state $|\phi_n\rangle$ (in which case a measurement of observable A yields the eigenvalue a_n) at time t is given by $|\langle\phi_n|\Psi(t)\rangle|^2$. That is, $\langle\phi_n|\Psi(t)\rangle$ is the probability amplitude for a system, in state $|\Psi(t)\rangle$, to be found in state $|\phi_n\rangle$ at time t.
4. The state of a system, immediately following a measurement of A that gave the value a_n, collapses to the state $|\phi_n\rangle$ (if a_n is degenerate, the state collapses to the subspace spanned by the degenerate states corresponding to the eigenvalue a_n).

We note that the eigenvalues of a hermitian operator are real; hence, the outcome of any measurement of an observable is a real number, as it should be. Further, for a hermitian operator, the eigenkets corresponding to different eigenvalues are necessarily orthogonal. In the case of a k-fold degeneracy, where k eigenkets correspond to the same eigenvalue, every ket in the k-dimensional subspace that the eigenkets span is an eigenket of A with the same eigenvalue. It is always possible to choose within this subspace a set of k eigenkets that are orthogonal to each other. By normalizing the eigenkets, it is always possible to choose the eigenkets $|\phi_1\rangle, |\phi_2\rangle, \ldots$ so as to form a complete orthonormal basis that spans the vector space of state vectors. Orthonormality means that $\langle\phi_i|\phi_j\rangle = \delta_{ij}$ where

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{1.7}$$

is the Kronecker delta, occasionally written as $\delta_{i,j}$ with a comma inserted between the indices if its absence could cause confusion. Completeness means that states $|\phi_1\rangle, |\phi_2\rangle, \ldots$ form a basis set: any state vector $|\Psi(t)\rangle \in \mathrm{V}$ can be expanded as

$$|\Psi(t)\rangle = \sum_n c_n(t)|\phi_n\rangle. \tag{1.8}$$

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If the basis is chosen to be an orthonormal one, i.e., if $|\phi_1\rangle, |\phi_2\rangle, \ldots$ form an orthonormal set, then for an arbitrary state $|\Psi(t)\rangle$,

$$\begin{aligned}|\Psi(t)\rangle &= \sum_n c_n(t)|\phi_n\rangle \Rightarrow \langle\phi_m\,|\Psi(t)\rangle = \sum_n c_n(t)\langle\phi_m|\phi_n\rangle = \sum_n c_n(t)\delta_{nm}\\ &= c_m(t) \Rightarrow |\Psi(t)\rangle = \sum_n \langle\phi_n\,|\Psi(t)\rangle\,|\phi_n\rangle = \sum_n |\phi_n\rangle\langle\phi_n\,|\Psi(t)\rangle\\ &\Rightarrow \sum_n |\phi_n\rangle\langle\phi_n| = 1. \end{aligned} \tag{1.9}$$

Equation (1.9) expresses mathematically the property of completeness of the orthonormal states $|\phi_1\rangle, |\phi_2\rangle, \ldots$. Note that $|\phi_n\rangle\langle\phi_n|$ is an operator: it acts on a ket to yield another ket, and the 1 on the right hand side (RHS) of Eq. (1.9) is the identity operator.

An important complete orthonormal set of states is formed by the eigenkets of the position operator $\mathbf{r}$,

$$\mathbf{r}|\mathbf{r}\rangle = \mathbf{r}|\mathbf{r}\rangle. \tag{1.10}$$

On the left hand side (LHS), $\mathbf{r}$ is the position operator, sometimes written as $\hat{\mathbf{r}}$ or $\mathbf{r}_{op}$ to emphasize that it is an operator, while $\mathbf{r}$ on the RHS is the eigenvalue of the position operator. The ket $|\mathbf{r}\rangle$ is the state of a particle with a well defined position $\mathbf{r}$. Since the operator $\mathbf{r}$ is hermitian, the states $|\mathbf{r}\rangle$ form a complete orthonormal set. Because $\mathbf{r}$ is continuous, the orthonormality and completeness of the states now read

$$\langle\mathbf{r}|\mathbf{r}'\rangle = \delta(\mathbf{r}-\mathbf{r}') \ \text{(orthonormality)}, \qquad \int |\mathbf{r}\rangle\langle\mathbf{r}|\, d^3r = 1 \ \text{(completeness)}. \tag{1.11}$$

$\delta(\mathbf{r}-\mathbf{r}')$ is the Dirac-delta function, defined as follows:

$$\delta(\mathbf{r}) = \begin{cases} 0 & \mathbf{r} \neq 0 \\ \infty & \mathbf{r} = 0 \end{cases} \tag{1.12}$$

and

$$\int \delta(\mathbf{r})d^3r = 1, \tag{1.13}$$

the integration being over all space. In one dimension, $\delta(x-x')$ is represented graphically as in Figure 1.3.

One important property of $\delta(\mathbf{r}-\mathbf{r}')$ is the sifting property,

$$\int f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')d^3r = f(\mathbf{r}'). \tag{1.14}$$

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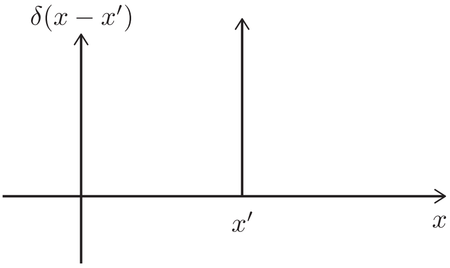

Figure 1.3 Dirac-delta function $\delta(x-x')$. It is zero for all values of x except for $x=x'$ where it is infinite. However, its integral over any interval containing x' is unity.

We also note that $\delta(\mathbf{r}-\mathbf{r}')=\delta(\mathbf{r}'-\mathbf{r})$ and $\delta(a\mathbf{r})=\delta(\mathbf{r})/|a|^d$, where d is the dimension of space: $d=3$ if $\mathbf{r}$ is a three-dimensional vector. A particularly useful representation of the Dirac-delta function is the following:

$$\delta(\mathbf{r})=\frac{1}{(2\pi)^3}\int e^{\pm i\mathbf{k}.\mathbf{r}}d^3k. \tag{1.15}$$

Another useful representation of the Dirac-delta function is

$$\delta(x)=d\theta(x)/dx \tag{1.16}$$

where $\theta(x)$ is the step function:

$$\theta(x)=\begin{cases}0 & x<0\\ 1 & x>0.\end{cases} \tag{1.17}$$

Note that $d\theta(x)/dx=0$ for $x\neq 0$, $d\theta(x)/dx=\infty$ for $x=0$, and the integral of $d\theta(x)/dx$ over any interval that includes $x=0$ is equal to 1.

Introducing a resolution of identity ($1=\int|\mathbf{r}\rangle\langle\mathbf{r}|d^3r$), the state vector $|\Psi(t)\rangle$ may be written as

$$|\Psi(t)\rangle=\int|\mathbf{r}\rangle\langle\mathbf{r}|\Psi(t)\rangle\,d^3r.$$

This is the continuous analog of the discrete case for which $|\Psi(t)\rangle=\sum_n|\phi_n\rangle\langle\phi_n|\Psi(t)\rangle$.

$|\langle\phi_n|\Psi(t)\rangle|^2$ has been interpreted as the probability for a particle in state $|\Psi(t)\rangle$ to be found at time t in state $|\phi_n\rangle$. By analogy, should $|\langle\mathbf{r}|\Psi(t)\rangle|^2$ be interpreted as the probability for a particle in state $|\Psi(t)\rangle$ to be found at time t in state $|\mathbf{r}\rangle$, i.e., to be at position $\mathbf{r}$ at time t? Two problems beset this interpretation:

(a) $|\Psi(t)\rangle$ and $|\phi_n\rangle$ are dimensionless ($\langle\Psi(t)|\Psi(t)\rangle=1$, $\langle\phi_n|\phi_n\rangle=1$); hence, $|\langle\phi_n|\Psi(t)\rangle|^2$ is dimensionless and can be interpreted as a probability. However,

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the orthonormality and completeness relations for states $|\mathbf{r}\rangle$, as expressed in Eq. (1.11), reveal that states $|\mathbf{r}\rangle$ have dimension $1/\text{Length}^{3/2}$. Thus, $|\langle \mathbf{r}\,|\Psi(t)\rangle|^2$ has dimension 1/Volume, and it cannot be interpreted as a probability; rather, it is more properly interpreted as a probability density.

(b) Suppose that a particle is in state $|\Psi(t)\rangle$ and a measurement is carried out to determine its position. No detector could ever pinpoint the location of a particle to exactly one point; the best a detector could do is to "click" whenever the particle is in some small volume d^3r surrounding the position $\mathbf{r}$. Assuming that $\langle \mathbf{r}\,|\Psi(t)\rangle$ does not change appreciably within the volume d^3r, the probability that the detector clicks should be proportional to $|\langle \mathbf{r}\,|\Psi(t)\rangle|^2 d^3r$. The constant of proportionality is determined by requiring that the probability of finding the particle somewhere in space be unity. Noting that

$$\int |\langle \mathbf{r}\,|\Psi(t)\rangle|^2 d^3r = \int d^3r \langle \mathbf{r}\,|\Psi(t)\rangle^* \langle \mathbf{r}\,|\Psi(t)\rangle = \int d^3r \langle \Psi(t)|\mathbf{r}\rangle\langle \mathbf{r}\,|\Psi(t)\rangle$$
$$= \langle \Psi(t)\,|\Psi(t)\rangle = 1\,,$$

the proportionality constant is seen to be 1. $\langle \mathbf{r}\,|\Psi(t)\rangle|^2 d^3r$ is thus interpreted as the probability for a particle in state $|\Psi(t)\rangle$ to be found at time t in the infinitesimal volume d^3r centered on $\mathbf{r}$. Comparing this with the probabilistic interpretation of $\Psi(\mathbf{r}, t)$ given in postulate I, the following identification is made:

$$\langle \mathbf{r}\,|\Psi(t)\rangle = \Psi(\mathbf{r}, t). \tag{1.18}$$

The state vector $|\Psi(t)\rangle$ may now be written as

$$|\Psi(t)\rangle = \int |\mathbf{r}\rangle\langle \mathbf{r}\,|\Psi(t)\rangle\, d^3r = \int \Psi(\mathbf{r}, t)|\mathbf{r}\rangle d^3r.$$

In other words, the wave function $\Psi(\mathbf{r}, t)$ is the component of state vector $|\Psi(t)\rangle$ along $|\mathbf{r}\rangle$.

In the $\mathbf{r}$-representation, the orthonormality of states $|\phi_1\rangle,\ |\phi_2\rangle, \ldots$ reads

$$\delta_{ij} = \langle \phi_i|\phi_j\rangle = \int \langle \phi_i|\mathbf{r}\rangle\langle \mathbf{r}|\phi_j\rangle d^3r = \int \phi_i^*(\mathbf{r})\, \phi_j(\mathbf{r})\, d^3r\,, \tag{1.19}$$

and their completeness is expressed as

$$1 = \sum_n |\phi_n\rangle\langle \phi_n| = \sum_n \iint |\mathbf{r}\rangle\langle \mathbf{r}|\phi_n\rangle\langle \phi_n|\mathbf{r}'\rangle\langle \mathbf{r}'|\, d^3r\, d^3r'$$
$$= \iint \sum_n \phi_n(\mathbf{r})\, \phi_n^*(\mathbf{r}')|\mathbf{r}\rangle\langle \mathbf{r}'|\, d^3r\, d^3r'.$$

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For the above to be true, it must be that

$$\sum_n \phi_n(\mathbf{r})\, \phi_n^*(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{1.20}$$

This expresses the completeness property in the **r**-representation.

We note that if operators A and B, representing two observables, commute ($AB = BA$), a complete set of states can be chosen so as to be simultaneous eigenstates of A and B; A and B may then be measured simultaneously.

So far, the fact that particles have spin has been ignored. To specify the state of a particle, its spin state must also be specified. For example, an electron has spin $s = 1/2$, and the z-component S_z of the spin operator has eigenvalues $+\hbar/2$ and $-\hbar/2$,

$$S_z\,|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle, \quad S_z\,|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle. \tag{1.21}$$

The spin-up state $|\uparrow\rangle$ is also denoted by $|1/2\rangle$, or $|+\rangle$, or α, while the spin-down state may also be written as $|-1/2\rangle$, or $|-\rangle$, or β. A general spin state, denoted by $|\chi\rangle$, is a linear combination of the basis states $|\uparrow\rangle$ and $|\downarrow\rangle$,

$$|\chi\rangle = a\,|\uparrow\rangle + b\,|\downarrow\rangle$$

where $a = \langle\uparrow|\chi\rangle$ and $b = \langle\downarrow|\chi\rangle$. If $|\chi\rangle$ is normalized ($\langle\chi|\chi\rangle = 1$), the probability of finding the spin up is $|a|^2$ and that of finding it down is $|b|^2$.

The spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ span a two-dimensional complex vector space, the spin space V_{spin}: they form an orthonormal basis for V_{spin},

$$\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1, \quad \langle\uparrow|\downarrow\rangle = 0, \quad |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = 1. \tag{1.22}$$

The above equations express, respectively, normalization, orthogonality, and completeness of the spin states. In general, for a particular spin s, the spin projection $\sigma = -s, -s+1, \ldots, s$; the spin space is a $(2s+1)$-dimensional complex vector space. The orthonormality and completeness relations are

$$\langle\sigma|\sigma'\rangle = \delta_{\sigma\sigma'}, \quad \sum_\sigma |\sigma\rangle\langle\sigma| = 1 \tag{1.23}$$

where $\sigma, \sigma' = -s, -s+1, \ldots, s$.

On the other hand, the states $|\phi_n\rangle$, which are eigenstates of a linear hermitian operator that depends on spatial coordinates, span a spatial vector space $\mathrm{V}_{\text{spatial}}$. The states $|\phi_\nu\rangle = |\phi_n\rangle \otimes |\sigma\rangle$, $\sigma = -s, -s+1, \ldots, s$ and $n = 1, 2, \ldots$ form an orthonormal basis for the direct product space $\mathrm{V} = \mathrm{V}_{\text{spatial}} \otimes \mathrm{V}_{\text{spin}}$, known as the Hilbert space. The state of a particle is a vector $|\Psi(t)\rangle \in \mathrm{V}$; hence, it can be

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expanded in the basis states,

$$|\Psi(t)\rangle = \sum_{n\sigma} c_{n\sigma}(t)\,|\phi_n\rangle \otimes |\sigma\rangle = \sum_{\nu} c_\nu(t)|\phi_\nu\rangle. \tag{1.24}$$

Here, ν is a collective index that specifies the spatial and spin quantum numbers. For example, four quantum numbers specify the eigenstates of a hydrogen atom: the principal quantum number n that determines the energy of the state, l which determines the value of L^2 (the square of the orbital angular momentum), m which determines the value of L_z (the z-component of the orbital angular momentum), and σ which is either ↑ or ↓. In this case $\nu = [nlm\sigma]$, while the index n in $|\phi_n\rangle$ stands for the spatial quantum numbers $[nlm]$. The ket $|\phi_\nu\rangle = |\phi_n\rangle \otimes |\sigma\rangle$, being a direct product of an orbital (spatial) state and a spin state, is called a spin orbital.

The orthonormality and completeness of the states $|\phi_\nu\rangle$ mean that

$$\langle\phi_\nu|\phi_{\nu'}\rangle = \langle\phi_n|\phi_{n'}\rangle\langle\sigma|\sigma'\rangle = \delta_{nn'}\delta_{\sigma\sigma'} = \delta_{\nu\nu'} \tag{1.25}$$

$$\sum_\nu |\phi_\nu\rangle\langle\phi_\nu| = \sum_n |\phi_n\rangle\langle\phi_n| \otimes \sum_\sigma |\sigma\rangle\langle\sigma| = 1_{\text{spatial}} \otimes 1_{\text{spin}} = 1. \tag{1.26}$$

Here, 1_{spatial} (1_{spin}) is the identity operator in V_{spatial} (V_{spin}), and 1 on the RHS is the identity operator in the Hilbert space (the direct product space).

So far, we have restricted the discussion to a one-particle system. We now consider a system comprised of N identical particles. Identical particles, such as electrons, are truly indistinguishable in quantum mechanics. The stationary states (eigenfunctions of the Hamiltonian H) of a system of N identical particles will be written as $\Psi(1, 2, \ldots, N)$, depending on the spatial and spin coordinates of the particles. Because of the indistinguishability of the particles, the Hamiltonian remains unchanged if any two particles are interchanged. This means that H commutes with P_{ij}, the permutation operator which interchanges particles i and j. It follows that the eigenfunctions of H can be chosen to be simultaneously eigenfunctions of P_{ij}. Denoting the eigenvalues of P_{ij} by λ, we can write

$$P_{ij}\Psi(1, \ldots, i, \ldots, j, \ldots, N) = \lambda\Psi(1, \ldots, i, \ldots, j, \ldots, N).$$

Applying P_{ij} to both sides of the above equation, and noting that $P_{ij}^2 = 1$, we obtain

$$\Psi(1, \ldots, i, \ldots, j, \ldots, N) = \lambda^2\Psi(1, \ldots, i, \ldots, j, \ldots, N).$$

Thus, $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$. For $\lambda = +1(-1)$, the wave function is symmetric (antisymmetric) under the exchange of coordinates (spatial and spin) of any two particles. In nature, particles with integral spin (0, 1, 2, ...), known as bosons, have symmetric wave functions under the exchange of the coordinates of two particles, and they obey Bose–Einstein statistics. On the other hand, particles with half integral spin (1/2, 3/2, ...), known as fermions, have antisymmetric wave functions

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under the exchange of the coordinates of two particles, and they obey Fermi–Dirac statistics. The Pauli exclusion principle is a direct consequence of this antisymmetry of the fermionic wave function. The last postulate of quantum mechanics follows.

(V) Wave function of a system of identical particles

Under the interchange of all coordinates (spatial and spin) of one particle with those of another, the wave function of a collection of identical particles must be symmetric if the particles are bosons, and antisymmetric if the particles are fermions:

$$\Psi(1,\ldots,j,\ldots,i,\ldots,N)=\begin{cases}\Psi(1,\ldots,i,\ldots,j,\ldots,N) & \text{Bosons}\\ -\Psi(1,\ldots,i,\ldots,j,\ldots,N) & \text{Fermions.}\end{cases} \tag{1.27}$$

We close this section by remarking that some exotic quasiparticles, known as anyons, which arise as excitations of a two-dimensional electron gas in a magnetic field, are believed to obey some fractional statistics, which are neither Bose–Einstein nor Fermi–Dirac statistics (Wilczek, 1982).

1.2 The harmonic oscillator

We briefly review the solution of the harmonic oscillator problem in quantum mechanics. For a particle of mass m confined to a harmonic potential, the Hamiltonian is given by

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2, \tag{1.28}$$

where ω is the oscillator frequency. We introduce two new operators

$$a=\left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x+\frac{i}{m\omega}p\right),\quad a^\dagger=\left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x-\frac{i}{m\omega}p\right). \tag{1.29}$$

Since x and p are hermitian, $a^\dagger$ is the adjoint of a, and vice versa. The operators x and $p=-i\hbar d/dx$ do not commute: $xp\neq px$. We define the commutator of any two operators A and B by

$$[A,B]=AB-BA. \tag{1.30}$$

By letting the commutator $[x,p]$ act on an arbitrary differentiable function $f(x)$, it is found that $[x,p]=i\hbar$. It follows that $[a,a^\dagger]=1$. In terms of the new operators,

$$H=\hbar\omega(N+1/2),\quad N=a^\dagger a. \tag{1.31}$$

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