

CAMBRIDGE TRACTS IN MATHEMATICS

General Editors

B. BOLLOBÁS, W. FULTON, A. KATOK,
F. KIRWAN, P. SARNAK, B. SIMON, B. TOTARO

195 A Universal Construction for Groups Acting Freely on Real Trees

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A Universal Construction for Groups Acting Freely on Real Trees

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to the memory of
KARL W. GRUENBERG
1928–2007

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Preface

In summer 2004, V. N. Remeslennikov, during a visit to Queen Mary and Westfield College, gave a series of three talks in which he outlined the construction of a class of groups $\mathcal{RF}(G)$, starting from the collection of (set-theoretic) functions $f : [0, \alpha] \rightarrow G$, where α is any non-negative real number and G is a given (discrete) group. Apparently, his main motivation was to imitate the construction of free groups in a continuous setting. He indicated that these new groups would have natural \mathbb{R} -tree actions associated with them, and he pointed out that it might be possible to study the centraliser of a hyperbolic element f in terms of (suitably defined) periods of f . However, no proofs were given.

Nevertheless, the picture emerging was felt to be interesting; the authors set out to try to fill in missing proofs, at first with the modest aim of establishing that the construction really produced groups. This task alone turned out to be rather difficult, leading to the development of a substantial body of cancellation theory (as given in Chapter 2 of the present book) before the actual proof that ‘reduced multiplication’ was associative could be given. By the time this task was accomplished (more than half a year later), the authors were already absorbed in what turned out to be a difficult but ultimately rewarding theory.

Now, several years further on, we present the fruits of our labour. To mention just a few highlights: the bounded subgroups of $\mathcal{RF}(G)$ are determined; it is shown that $\mathcal{RF}(G)$ (if non-trivial) is not generated by its elliptic elements and that the quotient of $\mathcal{RF}(G)$ by the span $E(G)$ of the elliptic elements has an isomorphism type depending at most on two cardinal numbers, the number of involutions in G as well as the cardinality of its complement. Moreover, the conjugacy relation for hyperbolic elements is characterised, thereby yielding a continuous analogue of the classical conjugacy theorem for free groups; cf. Theorem 1.3 in Magnus, Karrass, and Solitar [31].

Also, Remeslennikov's prediction concerning the centralisers of hyperbolic elements ultimately turns out to be substantially true, with some modification, but to prove this involves a considerable amount of work. Further, the last section of Chapter 10 contains the beginnings of a structure theory for $\mathcal{RF}(G)$ and its quotient $\mathcal{RF}(G)/E(G)$, while Chapter 4 explains our recent finding that \mathcal{RF} -groups and their associated \mathbb{R} -trees are universal (with respect to inclusion) for free \mathbb{R} -tree actions.

Something has been accomplished, yet much remains to be done. Nevertheless, as far as the case $\Lambda = \mathbb{R}$ is concerned the theory is beginning to shape nicely, despite the fact that there are still a large number of open problems (see Appendix B for a sample); thus it seemed a good idea, and the right time, to present our findings obtained so far in the hope of stimulating further research in what the authors feel is an exciting new area.