1
Introduction

In the first four sections we show how, starting with the usual description of free groups by means of reduced words, it is possible to arrive at a definition of the groups $\mathscr{RF}(G)$ and their associated $\mathbb{R}$-trees $\mathbf{X}_G$, which are the objects of study in this book. The final section summarises the contents of the following chapters.

1.1 Finite words and free groups

In constructing free groups, one may start from the collection of all finite *words*

$$w = x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_n}^{e_n}$$

over an alphabet $X \cup X^{-1}$, where X is some given set, $e_1, \dots, e_n \in \{1, -1\}$, and

$$X^{-1} = \{x^{-1} : x \in X\}$$

is a set in one-to-one correspondence with X via the map $x \mapsto x^{-1}$ such that $X \cap X^{-1} = \varnothing$. We extend this map to an involution of $X \cup X^{-1}$ by setting $(x^{-1})^{-1} = x$. A word w can be thought of as a function

$$\{1, 2, \dots, n\} \to X \cup X^{-1},$$

for some integer $n \geq 0$, the unique word of length 0 being the *empty word* ε. A word $w = x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_n}^{e_n}$ is called *reduced* if we have $x_{i_j}^{e_j} \neq x_{i_{j+1}}^{-e_{j+1}}$ for all indices j with $1 \leq j \leq n-1$, that is, if w does not contain a subword of the form $x_i^e x_i^{-e}$. Clearly, the empty word ε itself is reduced.

1.2 Words over a discretely ordered abelian group Λ

One can generalise the above set-up by taking an arbitrary discretely ordered abelian group[1] Λ, and considering 'infinite words' $w : [1,\alpha] \to X \cup X^{-1}$ for $\alpha \geq 0$, where

$$[1,\alpha] = \{\beta \in \Lambda : 1 \leq \beta \leq \alpha\}$$

and where 1 denotes the least positive element of Λ, the case $\alpha = 0$ corresponding to the empty word ε. This has indeed been done; see Myasnikov, Remeslennikov and Serbin [40]. In this setting the concept of reducedness still makes sense: a word w as above is *reduced*, if there does not exist $\beta \in [1,\alpha-1]$ such that $w(\beta+1) = w(\beta)^{-1}$. Clearly, the empty word ε is reduced. Let $R(\Lambda,X)$ be the set of all reduced words. We define the *inverse* of a word w on $[1,\alpha]$ as the function w^{-1} given on the same domain $[1,\alpha]$ by

$$w^{-1}(\beta) = w(\alpha-\beta+1)^{-1}, \quad 1 \leq \beta \leq \alpha.$$

One can check immediately that if w is reduced then so is w^{-1}.

The concatenation of two words u,v on domains $[1,\alpha]$ and $[1,\beta]$, respectively, is defined in a natural way as the word $u \circ v$ with domain $[1,\alpha+\beta]$ given by

$$(u \circ v)(\xi) = \left\{ \begin{array}{ll} u(\xi), & 1 \leq \xi \leq \alpha \\ v(\xi-\alpha), & \alpha+1 \leq \xi \leq \alpha+\beta \end{array} \right\} \quad (\xi \in [1,\alpha+\beta]).$$

In this situation one can define a partial multiplication (reduced concatenation) on $R(\Lambda,X)$ in a way that is analogous to multiplication in a free group. We first define, for $u,v \in R(\Lambda,X)$, $\mathrm{com}(u,v)$ to be the largest common initial segment of u and v, more precisely, $\mathrm{com}(u,v) = u|_{[1,\gamma]}$ with $\gamma \in \Lambda$ and $\gamma \geq 0$ such that

$$u(\xi) = v(\xi), \quad \xi \in [1,\gamma],$$

and either $\gamma = \min\{\alpha,\beta\}$ or $u(\gamma+1) \neq v(\gamma+1)$. The problem with this definition is, of course, that $\mathrm{com}(u,v)$ does not always exist, for which reason we shall only be able to define a partial multiplication on $R(\Lambda,X)$. Suppose that $w := \mathrm{com}(u^{-1},v)$ is defined. Then we can write $u^{-1} = w \circ u_1$, $v = w \circ v_1$, so that $u = u_1^{-1} \circ w^{-1}$, and we define the reduced product uv of the reduced words u and v by setting

$$uv = u_1^{-1} \circ v_1.$$

[1] By an ordered abelian group, we shall always mean a *totally ordered* abelian group.

Since u and v are reduced, so is uv. In this way, we obtain a partial multiplication on $R(\Lambda, X)$, which one can show is associative if it is defined; that is, if uv and vw are defined, then $(uv)w$ is defined if and only if $u(vw)$ is defined, in which case $(uv)w = u(vw)$. (Unfortunately, none of the elegant constructions of a free group that circumvent the need for establishing associativity work directly in this situation.)

Note that the *empty word* ε (corresponding to $\alpha = 0$) is a two-sided identity element, that is,

$$\varepsilon u = u = u\varepsilon, \quad u \in R(\Lambda, X).$$

Also, we have

$$uu^{-1} = \varepsilon = u^{-1}u, \quad u \in R(\Lambda, X).$$

Apart from the fact that reduced multiplication is only a partial operation, another marked difference from the free group case is that there can be words w with $w \neq \varepsilon$ but $w^2 = \varepsilon$.

Example 1.1 Let $\Lambda = \mathbb{Z}^2$ with right lexicographic ordering, so that the least positive element is $(1,0)$. Let $\alpha = (0,1)$ and fix $x \in X$. Define a word w on $[(1,0), \alpha]$ via

$$w(\beta) = \begin{cases} x, & \beta = (s,0),\ s \geq 1, \\ x^{-1}, & \beta = (s,1),\ s \leq 0. \end{cases}$$

Then w is reduced and non-trivial, and $w^2 = \varepsilon$.

There is also a notion of a *cyclically reduced* word: a word $w \in R(\Lambda, X)$ is cyclically reduced if $w(1) \neq w(\alpha)^{-1}$. Let

$$CDR(\Lambda, X) = \left\{ w \in R(\Lambda, X) : w = u \circ v \circ u^{-1} \text{ for some cyclically reduced word } v \right\}.$$

One can show that

$$CDR(\Lambda, X) = \left\{ w \in R(\Lambda, X) : w^2 \text{ is defined and } w^2 \neq \varepsilon \right\} \cup \left\{ \varepsilon \right\};$$

see Lemma 3.6 in [40].

We say that $G \subseteq CDR(\Lambda, X)$ is a subgroup of $CDR(\Lambda, X)$, if $u, v \in G$ implies that uv is defined and that $uv \in G$, if $u \in G$ implies that $u^{-1} \in G$, and if $\varepsilon \in G$. If G is a subgroup of $CDR(\Lambda, X)$, one can show that the function $L : G \to \Lambda$ given by $L(w) = \alpha$, where the domain of w is $[1, \alpha]$, and $L(\varepsilon) = 0$, is a Lyndon length function on G and gives rise to an action of G on a Λ-tree that is free and without inversions. (These terms are explained in Appendix A.) This generalises the fact that a free group, and so any subgroup, acts freely on its Cayley graph with respect to a basis; this graph is a tree. In fact, one can prove the following.

Theorem 1.2 *Let Λ be a discretely ordered abelian group. A group G acts freely and without inversions on a Λ-tree if and only if G is a subgroup of $CDR(\Lambda, X)$ for some set X.*

This is shown in [11]; the backward implication also appears in [40].

1.3 The case where Λ is densely ordered

At this stage, the question arises: can something analogous be done if instead we start from a *densely ordered* abelian group Λ? The first problem is that there is no longer a least positive element, so we replace a domain $[1,\alpha]$ with an interval $[0,\alpha]$ where $\alpha \geq 0$. A more serious problem, however, is that concatenation can no longer be defined as above. Our solution is to replace the set $X \cup X^{-1}$ by a (discrete) group G. Let

$$\mathscr{F}(\Lambda, G) := \bigcup_{\substack{\alpha\in\Lambda\\ \alpha\geq 0}} G^{[0,\alpha]} = \big\{f : [0,\alpha] \to G :\ \alpha \in \Lambda,\ \alpha \geq 0\big\}$$

be the set of all functions with values in G defined on an interval of Λ of the form $[0,\alpha]$ for some $\alpha \geq 0$. Concatenation is then replaced by an operation denoted $*$, the *star product*, defined as follows: if $f, g \in \mathscr{F}(\Lambda, G)$ are functions with domains $[0,\alpha]$ and $[0,\beta]$, respectively, then $f * g$ is the function given on the interval $[0,\alpha+\beta]$ of Λ via

$$(f*g)(\xi) = \left\{\begin{array}{ll} f(\xi), & 0 \leq \xi < \alpha \\ f(\alpha)g(0), & \xi = \alpha \\ g(\xi-\alpha), & \alpha < \xi \leq \alpha+\beta \end{array}\right\} \quad (\xi \in [0,\alpha+\beta]).$$

The function $\mathbf{1}_G$ defined on the interval $[0,0] = \{0\}$ by $\mathbf{1}_G(0) = 1_G$ (where 1_G is the identity element of G) is a two-sided identity element with respect to the star operation; that is, we have

$$f * \mathbf{1}_G = f = \mathbf{1}_G * f, \quad f \in \mathscr{F}(\Lambda, G).$$

We also have a notion of the *formal inverse* f^{-1} of a function $f \in \mathscr{F}(\Lambda, G)$: if f is defined on the domain $[0, \alpha]$ then f^{-1} is the function given on the same interval $[0, \alpha]$ by

$$f^{-1}(\xi) = \big(f(\alpha - \xi)\big)^{-1}, \quad 0 \leq \xi \leq \alpha.$$

In this setting there is also a notion of a reduced function, which necessarily needs to be somewhat more elaborate. A function $f \in \mathscr{F}(\Lambda, G)$ defined on the interval $[0, \alpha]$ of Λ is called *reduced* if, for each point $\xi_0 \in (0, \alpha)$ with $f(\xi_0) = 1_G$ and every element $\varepsilon \in \Lambda$ with $0 < \varepsilon \leq \min\{\alpha - \xi_0, \xi_0\}$, there exists some $\delta \in \Lambda$ such that $0 < \delta \leq \varepsilon$ and such that $f(\xi_0 + \delta) \neq \big(f(\xi_0 - \delta)\big)^{-1}$. The set of all reduced functions in $\mathscr{F}(\Lambda, G)$ is denoted by $\mathscr{RF}(\Lambda, G)$. Given a function $f : [0, \alpha] \to G$ in $\mathscr{F}(\Lambda, G)$, let us call an ε-neighbourhood

$$[\xi_0 - \varepsilon, \xi_0 + \varepsilon] \subseteq [0, \alpha]$$

of a point $\xi_0 \in (0, \alpha)$, with $f(\xi_0) = 1_G$, a *cancelling neighbourhood around* ξ_0 if $f(\xi_0 - \delta) = \big(f(\xi_0 + \delta)\big)^{-1}$ for all $0 < \delta \leq \varepsilon$. Then we can say that a function $f \in \mathscr{F}(\Lambda, G)$ as above is reduced if and only if there does not exist a cancelling neighbourhood around any interior point of the domain $[0, \alpha]$ of f satisfying $f(\xi_0) = 1_G$.

For $u, v \in \mathscr{F}(\Lambda, G)$, an analogue of $\mathrm{com}(u, v)$ can be defined and, if the element $\mathrm{com}(u^{-1}, v) =: w$ exists, so that $u^{-1} = w * u_1$ and $v = w * v_1$, we may define the reduced product uv of u and v by $uv = u_1^{-1} * v_1$. This gives a partial multiplication on $\mathscr{F}(\Lambda, G)$ that is associative when defined. It can also be shown that the product of two reduced functions, when it exists, is again reduced.

1.4 The case where $\Lambda = \mathbb{R}$

In this book, we shall confine our attention to the case where $\Lambda = \mathbb{R}$, taking the view that this is already quite difficult to deal with (in particular, the proof of the associativity of reduced multiplication is non-trivial). In this case (reduced) multiplication is always defined, and we obtain a group denoted by $\mathscr{RF}(G)$, with the formal inverse of a reduced function f acting as a two-sided inverse of f and with $\mathbf{1}_G$ as the neutral element.

There is a construction of an $\mathbb{R}$-tree $\mathbf{X}_G$ on which $\mathscr{RF}(G)$ acts with point stabilisers isomorphic to G. More precisely, by definition, each element f of $\mathscr{RF}(G)$ has a real number $L(f)$ assigned to it, namely the length α of its domain $[0, \alpha]$; it is not hard to see that the function $L : \mathscr{RF}(G) \to \mathbb{R}$ defined in this way is a Lyndon length function. It follows that $\mathscr{RF}(G)$ has a canonical action by isometries on an $\mathbb{R}$-tree

$$\mathbf{X}_G = (X_G, d)$$

with a distinguished base-point x_0 such that $L_{x_0} = L$, where L_{x_0} is the displacement function

$$L_{x_0}(f) = d(x_0, f x_0), \quad f \in \mathscr{RF}(G)$$

associated with this action and such that

$$G_0 := \mathrm{stab}_{\mathscr{RF}(G)}(x_0) = \big\{ f \in \mathscr{RF}(G) : L(f) = 0 \big\} \cong G.$$

It turns out that $\mathbf{X}_G$ is always metrically complete and that the action of $\mathscr{RF}(G)$ on $\mathbf{X}_G$ is transitive.

As always in such situations, the action of $\mathscr{RF}(G)$ on $\mathbf{X}_G$ leads to a classification of the elements of $\mathscr{RF}(G)$ according to whether they are elliptic (that is, have a fixed point) or hyperbolic (that is, act as a fixed-point free isometry). Hyperbolic elements have some local geometry associated with them, leading, in particular, to another type of length function on $\mathscr{RF}(G)$: if $f \in \mathscr{RF}(G)$ is hyperbolic then there exists an isometric copy $A_f \subseteq \mathbf{X}_G$ of the real line (the so-called *axis* of f) such that f acts on A_f as a non-trivial translation; in particular, hyperbolic elements have infinite order. The translation length of a hyperbolic element f along its axis A_f is called the *hyperbolic length* of f, denoted $\ell(f)$, and ℓ is extended to the whole of $\mathscr{RF}(G)$ by setting $\ell(f) = 0$ for an elliptic function f.

With a view to investigating further the action of $\mathscr{RF}(G)$, we shall introduce and study an analogue of cyclic reduction in free groups; this allows us, among other things, to characterize hyperbolic elements in a purely algebraic way and to compute hyperbolic length in terms of the length function L.

It follows from the transitivity of the action that the set of elliptic elements in $\mathscr{RF}(G)$ coincides with the union of all conjugates of G_0; in particular, $\mathscr{RF}(G)$ has torsion if and only if the group G has. We will establish a stronger result to the effect that a subgroup of $\mathscr{RF}(G)$ is bounded (with respect to the length function L) if and only if it is conjugate to a subgroup of G_0. This result shows in particular that every finite subgroup of $\mathscr{RF}(G)$ is conjugate to a subgroup of G_0, a result reminiscent of the bounded subgroup theorem for free products with amalgamation; see, for instance, Theorem 8 of Chapter I in Serre [45]. It also follows that the trivial group $\{\mathbf{1}_G\}$ is the only bounded subnormal subgroup of $\mathscr{RF}(G)$.

1.5 Contents of the book

We now turn to a discussion of individual chapters.

Chapters 2 *and* 3. Here we give the basic definitions, introduce the groups $\mathscr{RF}(G)$, and develop some cancellation theory needed for (among other things) a proof of the associativity of reduced multiplication. We then study the geometry associated with $\mathscr{RF}(G)$ via its action on the $\mathbb{R}$-tree $\mathbf{X}_G$ and the classification of the group elements effected by this action, covering the ground indicated in Section 1.4 above and more.

Chapter 4. This chapter reflects a rather exciting new development, reporting on the authors' recent discovery of two basic *embedding theorems*. We show that a group G acting freely and without inversions on a Λ-tree $\mathbf{X}$ (for an arbitrary ordered abelian group Λ) can be embedded into a group $\hat{G}$, acting freely, without inversions, and *transitively* on the Λ-tree $\hat{\mathbf{X}}$, which isometrically and G-equivariantly embeds $\mathbf{X}$. This result is of considerable independent interest, in particular shedding new light on the class of infinitely generated $\mathbb{R}$-free groups.

We then proceed to discuss a second, more specialised, embedding theorem concerning free and transitive $\mathbb{R}$-tree actions: we show that a group G acting freely and transitively on an $\mathbb{R}$-tree $\mathbf{X}$ can be embedded into $\mathscr{RF}(H)$ for some suitable group H such that $\mathbf{X}$ embeds isometrically and G-equivariantly into $\mathbf{X}_H$, the $\mathbb{R}$-tree canonically associated with $\mathscr{RF}(H)$.

Combining these two results we infer that $\mathscr{RF}$-groups and their associated $\mathbb{R}$-trees are in fact *universal* (with respect to inclusion) for free $\mathbb{R}$-tree actions.

Chapter 5. Very little is known at present concerning homomorphisms involving $\mathscr{RF}$-groups. In this chapter a certain homomorphism

$$e_g : \mathscr{RF}(G) \to \mathbb{R}$$

is defined for each element $g \in G$ by means of Lebesgue measure theory. The construction of these maps e_g is analogous to and inspired by the exponent sum maps of a free group relative to a basis element. By construction, the elliptic elements of $\mathscr{RF}(G)$ are contained in the kernel of e_g for every g, and if $g \in G$ is not an involution then the corresponding map e_g is surjective; this shows in particular that if G is not an elementary abelian 2-group then $\mathscr{RF}(G)$ is not generated by its elliptic elements. For G an elementary abelian 2-group, the question remains open at this stage since all exponent sums of G are trivial.

The problem is taken up and resolved in Chapter 9 as part of the theory of test functions (see below).

Chapter 6. In this chapter we explore various aspects of functoriality of the $\mathscr{RF}$-construction. The most striking result obtained here is that if two groups G and H have the same (cardinal) number of involutions and the same number of non-involutions then we have

$$\mathscr{RF}(G)/E(G) \cong \mathscr{RF}(H)/E(H),$$

where $E(G)$ is the subgroup of $\mathscr{RF}(G)$ generated by the elliptic elements, that is, the normal closure of G_0. With slight imprecision the last result may be rephrased as follows.

The isomorphism type of the group $\mathscr{RF}(G)/E(G)$ depends only on the two cardinal numbers $|\mathrm{Inv}(G)|$ and $|G - \mathrm{Inv}(G)|$.

The proof of this surprising and rather deep lying *rigidity result* is long and somewhat technical. However, the techniques developed in this chapter also allow us to obtain at least a partial result concerning the automorphism group of the quotient group $\mathscr{RF}(G)/E(G)$; see Proposition 6.7 and Corollary 6.8.

Chapter 7. A guiding principle when investigating $\mathscr{RF}$-groups appears to be the following.

Hyperbolic elements of a non-trivial $\mathscr{RF}$-group behave analogously to the non-trivial elements of a (large) free group.

This principle manifests itself for instance in the *conjugacy theorem for hyperbolic elements* established in Chapter 7, which (except for its proof) is an exact continuous analogue of the corresponding result for free groups. We also show there that the centraliser of a hyperbolic element $f \in \mathscr{RF}(G)$ has index at most 2 in the normaliser of the infinite cyclic group $\langle f \rangle$ in $\mathscr{RF}(G)$.

Chapter 8. It is easy to see that, for a non-trivial element $g \in G_0$, we have

$$C_{\mathscr{RF}(G)}(g) = C_{G_0}(g).$$

Consequently the centralisers of elliptic elements in $\mathscr{RF}(G)$ are determined, up to isomorphism, by the isomorphism types of the centralisers in the group G itself; hence, in general (that is, without restricting the structure of G), nothing more can be said here.

The situation is very different, and much more interesting, for hyperbolic elements, and the present chapter provides a penetrating study of their centralisers. We establish a criterion characterising those hyperbolic elements whose centraliser is cyclic and obtain considerable insight into the centraliser structure in the general case; in particular, we show that centralisers of hyperbolic elements are abelian and relatively 'small', in that they always embed into the additive reals. As suggested by Remeslennikov, the centraliser $C_{\mathscr{RF}(G)}(f)$ for hyperbolic f is controlled by (a subset of) the periods of the function f; the reader is referred to Chapter 8 for details.

As an application of the main result of that chapter (Theorem 8.16), we show that $\mathscr{RF}$-groups enjoy an analogue of the *centraliser partition property* of free groups: the binary relation $\leftrightarrow$ given by

$$f \leftrightarrow g \;:\Longleftrightarrow\; f \text{ and } g \text{ commute}$$

is an equivalence relation on the set

$$\mathscr{RF}(G) - \bigcup_{t \in \mathscr{RF}(G)} tG_0t^{-1}$$

of hyperbolic elements of $\mathscr{RF}(G)$; see Proposition 8.23 and Corollary 8.24. This result provides a further illustration of the philosophy concerning hyperbolic elements expressed above. As another application of Theorem 8.16, we show that $\mathscr{RF}$-groups do not contain non-trivial soluble normal subgroups. A completely different approach to this last result, using the theory of test functions, is given in Chapter 10.

Chapters 9 *and* 10. These two chapters provide an introduction to the theory of test functions and its applications, as developed originally in Müller [36] and Müller and Schlage-Puchta [38]. Roughly speaking, a test function is a mapping $f : [0,\alpha] \to G$ of positive length $L(f) = \alpha$, such that f does not look locally like its own inverse. More precisely, we require that there do not exist $\varepsilon > 0$ and points $\xi_1, \xi_2 \in (0,\alpha)$ such that

$$f(\xi_1 + \eta) = f^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon.$$

Test functions do in fact always exist; for instance, the function f_0 of length 1 given by

$$f_0(\xi) = \left\{ \begin{matrix} x, & \xi^2 \in \mathbb{Q} \\ 1_G, & \xi^2 \notin \mathbb{Q} \end{matrix} \right\} \quad (0 \le \xi \le 1),$$

where x is any non-trivial element of G, can be shown to be a test function; see Section 9.3. Test functions are automatically (cyclically) reduced and give rise to a further class of homomorphisms $\mathscr{RF}(G) \to \mathbb{R}$. Roughly speaking, given a test function $f \in \mathscr{RF}(G)$ of length $\alpha > 0$, the idea is to compare ('test') functions $g \in \mathscr{F}(G)$ locally against f and f^{-1}, in this way obtaining two sets $\mathscr{M}_f^+(G), \mathscr{M}_f^-(G) \subseteq (0, L(g))$. To be more explicit, we set

$$\mathscr{M}_f^{\pm}(g) := \Big\{ \xi \in (0, L(g)) : \exists \varepsilon > 0,\ \exists \xi' \in (0, \alpha) \text{ such that } g(\xi + \eta) = f^{\pm}(\xi' + \eta) \text{ for all } |\eta| < \varepsilon \Big\},$$

observing that $\mathscr{M}_f^+(g)$ and $\mathscr{M}_f^-(g)$ are open sets and thus Lebesgue measurable, and define a function $\lambda_f : \mathscr{RF}(G) \to \mathbb{R}$ by

$$\lambda_f(g) = \mu\big(\mathscr{M}_f^+(g)\big) - \mu\big(\mathscr{M}_f^-(g)\big),$$

where μ denotes Lebesgue measure. We show that λ_f is a surjective homomorphism whose kernel contains $E(G)$, in this way demonstrating in particular that $\mathscr{RF}(G)$ is never generated by its elliptic elements; see Theorem 9.8 and Corollary 9.9.

A second important idea introduced in Chapter 9 is that of *local compatibility* and *incompatibility*. Roughly speaking, given functions $f : [0, \alpha] \to G$ and $g : [0, \beta] \to G$, we say that f and g are locally compatible if f looks locally like g or g^{-1}. To be more precise, f and g as above are termed locally compatible if there exist $\varepsilon > 0$ and points $\xi \in (0, \alpha)$, $\zeta \in (0, \beta)$ such that either

$$f(\xi + \eta) = g(\zeta + \eta), \quad |\eta| < \varepsilon,$$

or

$$f(\xi + \eta) = g^{-1}(\zeta + \eta), \quad |\eta| < \varepsilon.$$

If f and g both have positive length but are not locally compatible then they are called locally incompatible. Locally incompatible functions have no cancellation against each other, and if f, g are locally incompatible then so are f^{-1} and g as well as f^{-1} and g^{-1}.

We call a subgroup $\mathscr{H} \leq \mathscr{RF}(G)$ *hyperbolic* if the set $\mathscr{H} - \{\mathbf{1}_G\}$ consists entirely of hyperbolic elements. As a further application of test function theory (as developed so far), in Section 9.6 we show among other things that the family of centralisers $\{C_{\mathscr{RF}(G)}(f_\sigma)\}_{\sigma \in S}$ corresponding to a family $\{f_\sigma\}_{\sigma \in S}$ of pairwise locally incompatible test functions generates a hyperbolic subgroup of $\mathscr{RF}(G)$ isomorphic to the free product $\ast_{\sigma \in S} C_{\mathscr{RF}(G)}(f_\sigma)$; see Corollary 9.24.

The most striking applications of test function theory to date, however, stem from a rather deep result (Theorem 10.1) asserting the existence of large families of pairwise locally incompatible test functions with prescribed centraliser: *given a non-trivial group G and a proper subgroup $\Lambda \leq (\mathbb{R}, +)$, there exists a family $\mathfrak{F}$ of pairwise locally incompatible test functions in $\mathscr{RF}(G)$ such*