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978-1-107-02464-9 — Microhydrodynamics, Brownian Motion, and Complex Fluids
Michael D. Graham
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1 Kinematics, Balance Equations, and Principles of Stokes Flow

The focus of this book is the dynamics of fluids at small scales and of small objects (e.g., particles, cells, macromolecules) suspended in fluids. As we will see, such suspensions or solutions can have nontrivial dynamical and rheological behavior: i.e., in this regard they are *complex fluids*. Small is relative, of course, and we exclusively consider systems that are not so small that atomistic details of the fluid or objects are important; in particular, we treat the fluid as a continuum to which we can assign properties at every spatial position $\boldsymbol{x}$. For liquids, the continuum approximation is broadly valid for scales of about 1 nanometer (nm) and larger (a water molecule has a size of about 0.2 nm). One way that we know this is through molecular simulations (Schmidt & Skinner 2003, 2004), which show agreement with, for example, the continuum prediction for the drag force on a moving sphere even when the sphere is only several solvent atoms across. In the first several chapters of this book, we will only concern ourselves with the behavior of fluids, and particles within fluids, in the absence of thermal fluctuations. This behavior is governed by the classical equations of continuum mechanics, which are the starting point of the chapter. After reviewing these here, the governing equations for Newtonian fluids are introduced. Our ultimate focus is the Stokes equation, which governs fluid motions when the inertia of the fluid is negligible compared to viscous stresses.

1.1 Kinematics of Continua

1.1.1 Velocity Fields and the Velocity Gradient

Under the continuum approximation, a material can take on a velocity $\boldsymbol{v}$ (which may vary with time t) at every position $\boldsymbol{x}$: i.e., we have a velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$. We further

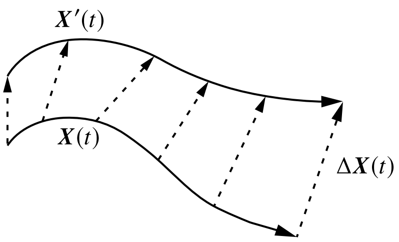

Figure 1.1 Motions of material points $\boldsymbol{X}(t)$ and $\boldsymbol{X}'(t)$, with the material line $\Delta\boldsymbol{X}(t) = \boldsymbol{X}'(t) - \boldsymbol{X}(t)$ (dashed) shown at several time instants.

assume that this field is differentiable except possibly at interfaces between different materials or phases. Now, consider two neighboring *material points* $\boldsymbol{X}$ and $\boldsymbol{X}'$, which by definition move with the instantaneous velocity of the material at their respective positions, as illustrated in Figure 1.1. Thus

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{v}(\boldsymbol{X}, t), \quad \frac{d\boldsymbol{X}'}{dt} = \boldsymbol{v}(\boldsymbol{X}', t).$$

To understand how the material deforms as it flows with velocity $\boldsymbol{v}(\boldsymbol{x}, t)$, we consider the time evolution of a "material line" $\Delta\boldsymbol{X} = \boldsymbol{X}' - \boldsymbol{X}$ connecting points $\boldsymbol{X}$ and $\boldsymbol{X}'$. We can then write

$$\frac{d\Delta\boldsymbol{X}}{dt} = \boldsymbol{v}(\boldsymbol{X}', t) - \boldsymbol{v}(\boldsymbol{X}, t).$$

Now we take the distance between $\boldsymbol{X}$ and $\boldsymbol{X}'$ to be small, so that

$$\boldsymbol{v}(\boldsymbol{X}', t) = \boldsymbol{v}(\boldsymbol{X}, t) + \Delta\boldsymbol{X} \cdot \boldsymbol{\nabla}\boldsymbol{v}(\boldsymbol{X}, t) + O(|\Delta\boldsymbol{X}|^2),$$

where $\boldsymbol{\nabla}\boldsymbol{v}(\boldsymbol{X}, t)$ is the *velocity gradient* evaluated at the material point $\boldsymbol{X}$. Combining these two equations yields that

$$\frac{d\Delta\boldsymbol{X}}{dt} = \Delta\boldsymbol{X} \cdot \boldsymbol{\nabla}\boldsymbol{v}, \tag{1.1}$$

or equivalently

$$\frac{d\Delta\boldsymbol{X}}{dt} = \mathbf{L} \cdot \Delta\boldsymbol{X}, \tag{1.2}$$

where $\mathbf{L} = \boldsymbol{\nabla}\boldsymbol{v}^{\mathrm{T}}$. Therefore, all information about the deformation of an infinitesimal line connecting two neighboring points in the fluid, is contained in the velocity gradient tensor.[1] Note that we use the convention[2] that in Cartesian coordinates

$$(\boldsymbol{\nabla}\boldsymbol{v})_{ij} = \frac{\partial}{\partial x_i} v_j.$$

Flows in which the velocity gradient is independent of position are called *linear flows*, because the velocity is a linear function of position:

$$\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}_0 = \mathbf{L} \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \tag{1.3}$$

where $\boldsymbol{v}_0$ is a constant uniform velocity and $\boldsymbol{x}_0$ is a constant position. By appropriate choice of reference frame, we can always take $\boldsymbol{v}_0$ and $\boldsymbol{x}_0$ to be zero. If in addition $\boldsymbol{\nabla}\boldsymbol{v}$ is independent of position and time, then (1.1) is a simple linear constant coefficient equation and the evolution of $\Delta\boldsymbol{X}$ is completely determined by the eigenvalues and eigenvectors of $\boldsymbol{\nabla}\boldsymbol{v}$. We will exclusively consider incompressible flows, which satisfy

[1] See Section A.1 for a brief summary of vector and tensor notation as used in this book.
[2] This convention is common in the fluid mechanics literature but is not universal. In much of the continuum mechanics literature, e.g. Malvern (1969), Gonzalez & Stuart (2008), $(\boldsymbol{\nabla}\boldsymbol{v})_{ij}$ is defined as $\frac{\partial v_i}{\partial x_j}$. This is the transpose of what we use.

$\nabla \cdot \boldsymbol{v} = 0$ due to mass conservation (see Section 1.2.1), so the eigenvalues λ_i of $\nabla \boldsymbol{v}$ must sum to zero. Equivalently,

$$\mathrm{tr}\, \nabla \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = \sum_{i=1}^{3} \lambda_i = 0.$$

We will often make use of the decomposition of the velocity gradient tensor into the symmetric *strain rate* or *deformation rate* tensor $\mathbf{E}$ and antisymmetric *vorticity* tensor[3] $\mathbf{W}$:

$$\nabla \boldsymbol{v} = \mathbf{E} + \mathbf{W},$$

where

$$\mathbf{E} = \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}\right)$$

and

$$\mathbf{W} = \frac{1}{2}\left(\nabla \boldsymbol{v} - \nabla \boldsymbol{v}^{\mathrm{T}}\right).$$

The vorticity *vector* $\boldsymbol{w}$ is given by

$$\boldsymbol{w} = \nabla \times \boldsymbol{v} \tag{1.4}$$

and is related to the local angular velocity of the fluid, $\boldsymbol{\omega}$, by the simple expression

$$\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{w}. \tag{1.5}$$

These quantities are related to the vorticity tensor as follows:

$$\mathbf{W} = \boldsymbol{\epsilon} \cdot \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\epsilon} \cdot \boldsymbol{w}, \tag{1.6}$$

where $\boldsymbol{\epsilon}$ is the Levi–Civita symbol, whose properties are summarized in Appendix A.1.2.

If $\mathbf{E} = \mathbf{0}$ at some point in the fluid, then an infinitesimal volume of material at that point, a *fluid element*, is undergoing rigid rotation: there is no stretching of material lines within that volume. On the other hand, if $\mathbf{W} = \mathbf{0}$, then the fluid element is undergoing stretching without any rotation. Now $\nabla \boldsymbol{v}$ is symmetric, so its eigenvectors are orthogonal, forming a coordinate system in which $\nabla \boldsymbol{v}$ can be written

$$\nabla \boldsymbol{v} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Defining the extension rate as $\dot{\epsilon}$ (> 0), important special cases include the following:

- Uniaxial extension: $\lambda_1 = \dot{\epsilon}, \lambda_2 = \lambda_3 = -\dot{\epsilon}/2$,
- Biaxial extension: $\lambda_1 = \lambda_2 = \dot{\epsilon}, \lambda_3 = -2\dot{\epsilon}$,
- Planar extension: $\lambda_1 = -\lambda_2 = \dot{\epsilon}, \lambda_3 = 0$.

[3] Again, the convention in the fluid mechanics literature is different than in the continuum mechanics literature, where $\mathbf{W} = (\mathbf{L} - \mathbf{L}^{\mathrm{T}})/2$ and is often called the *spin* tensor (Malvern, 1969).

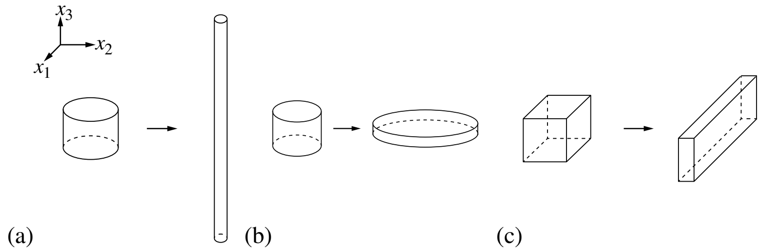

Figure 1.2 Deformation of a fluid element in (a) uniaxial extension, (b) biaxial extension, and (c) planar extension.

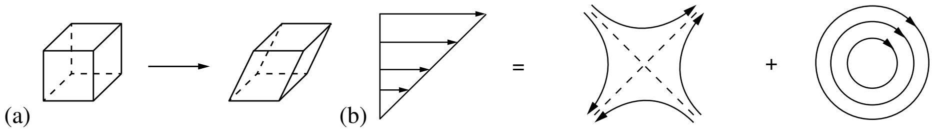

Figure 1.3 (a) Deformation of a volume of fluid in simple shear. (b) Decomposition of the velocity field of simple shear into equal parts of planar extension and rigid rotation.

In these cases, material lines will either shrink or grow exponentially in time, depending on their initial orientation relative to the eigenvectors of $\nabla\boldsymbol{v}$. Figure 1.2 illustrates how fluid elements evolve in these flows.

Simple shear flow, where $\boldsymbol{v} = \dot{\gamma} y \boldsymbol{e}_x$, is another important special case that deserves particular attention. In this case, illustrated in Figure 1.3(a), elements simply move in the x-direction in a straight line, the eigenvalues of $\nabla\boldsymbol{v}$ are all zero, and an arbitrarily oriented material line stretches linearly with time. In Cartesian coordinates,

$$\nabla\boldsymbol{v} = \begin{bmatrix} 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and thus

$$\mathbf{E} = \frac{1}{2}\begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{W} = \frac{1}{2}\begin{bmatrix} 0 & -\dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From (1.4) and (1.5),

$$\boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\dot{\gamma} \end{bmatrix}.$$

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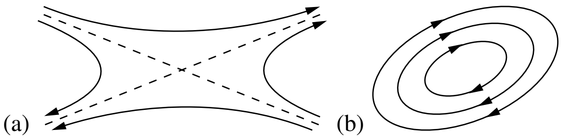

Figure 1.4 Paths of fluid elements for perturbations of simple shear with (a) $\alpha > 0$ and (b) $\alpha < 0$. The case $\alpha = 0$ is simple shear as shown in Figure 1.3.

Letting $||\cdot||$ denote the Frobenius norm when applied to a second-order tensor,[4] observe that in this case $||\mathbf{E}|| = ||\mathbf{W}||$. If the velocity gradient were composed entirely of $\mathbf{E}$, then fluid elements would undergo planar extension, compressing along the line $x = -y$, the "compressional axis" and stretching along the line $x = y$, the "extensional axis." On the other hand, if the velocity gradient were composed entirely of $\mathbf{W}$, then fluid elements would just rotate clockwise. The overall deformation that occurs during simple shear is an equal superposition of these deformations, as illustrated in Figure 1.3(b), in which the stretching due to $\mathbf{E}$ is just balanced by the rotation due to $\mathbf{W}$, and fluid elements stretch linearly in time, tilting toward the flow direction but without "tumbling." This result is a simple consequence of the fact that the particle paths are all straight lines.

Consider now a small change in the velocity gradient from the simple shear case (Fuller & Leal 1981):

$$\nabla \boldsymbol{v} = \dot{\gamma} \begin{bmatrix} 0 & \alpha & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.7}$$

Now

$$\mathbf{E} = \frac{1}{2}\dot{\gamma} \begin{bmatrix} 0 & 1+\alpha & 0 \\ 1+\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{W} = \frac{1}{2}\dot{\gamma} \begin{bmatrix} 0 & \alpha-1 & 0 \\ 1-\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Holding $\dot{\gamma}$ constant and restricting α to the range $-1 \leq \alpha \leq 1$, there are three cases as shown in Figure 1.4. If $\alpha = 0$, then the flow is simple shear. If $\alpha > 0$, then strain dominates over vorticity, $||\mathbf{E}|| > ||\mathbf{W}||$, the eigenvalues of $\nabla \boldsymbol{v}$ are real, particle paths are hyperbolas, and a material line stretches exponentially fast – in the limiting case $\alpha = 1$, the vorticity vanishes, and flow is pure planar extension, with compressional axis along the line $x = -y$ and extensional axis along $x = y$. On the other hand, if $\alpha < 0$, vorticity dominates, $||\mathbf{E}|| < ||\mathbf{W}||$, and particle paths are ellipses – an individual fluid element will oscillate in length. Accordingly, $\nabla \boldsymbol{v}$ has a pair of purely imaginary eigenvalues. (There is always one zero eigenvalue, independent of α, because there is no motion in the z direction.) When $\alpha = -1$, the strain rate vanishes and the flow is rigid rotation – particle paths are circles and material lines rotate without any change in length. Thus

[4] See Appendix A.1.

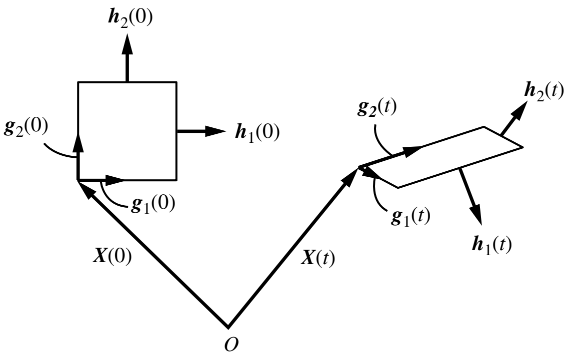

Figure 1.5 Basis vectors $\boldsymbol{g}_i(t)$ attached to a material point $\boldsymbol{X}(t)$ and evolving as material lines. At $t = 0$ (left) these vectors define the edges of a cube, only two dimensions of which are shown, that is deformed at time t to a parallelepiped. The reciprocal basis vectors $\boldsymbol{h}_i(t)$ define the faces of the parallelepiped.

we see that simple shear is a delicate special case, a point that should be remembered because it is so often used as a model flow.

1.1.2 Deformation Tensors

For time-independent linear flows, the eigenvalues and eigenvectors of $\nabla \boldsymbol{v}$ contain all the information needed to determine how material lines stretch and thus how the material deforms. For more complex flow fields, however, we require a more general formalism to characterize deformations (Malvern 1969, Bird, Armstrong, & Hassager 1987, Segel 1987, Gonzalez & Stuart 2008, Morozov & Spagnolie 2015). Consider a set of basis vectors $\boldsymbol{g}_i$ that are attached to a material element $\boldsymbol{X}(t)$. At $t = 0$, the basis vectors correspond to the Cartesian basis vectors: $\boldsymbol{g}_i(\boldsymbol{X}(0), 0) = \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the ith Cartesian unit basis vector. Thus the $\boldsymbol{g}_i$ are the edges of the parallelepiped that begins as cubic material volume attached to $\boldsymbol{X}(t)$. These basis vectors will be taken to evolve as material lines, as shown in Figure 1.5: i.e.,

$$\frac{d\boldsymbol{g}_i(\boldsymbol{X}(t), t)}{dt} = \mathbf{L}(\boldsymbol{X}(t), t) \cdot \boldsymbol{g}_i(\boldsymbol{X}(t), t). \tag{1.8}$$

Note that $\mathbf{L}$ is time dependent in general. These vectors form a basis that is *codeforming* with the material. Recall that at $t = 0$, the $\boldsymbol{g}_i$ are tangent to the Cartesian coordinate lines. If we take these coordinate lines to be embedded in the material, moving and deforming with it, then because the $\boldsymbol{g}_i$ evolve as material lines, they will each be tangent to the corresponding coordinate line at time t; basis vectors that are parallel to coordinate lines are said to be *contravariant* (Aris 1989).

The solution to (1.8) can be written as

$$\boldsymbol{g}_i(\boldsymbol{X}(t), t) = \mathbf{F}(\boldsymbol{X}(t), t) \cdot \boldsymbol{g}_i(\boldsymbol{X}(0), 0), \tag{1.9}$$

where $\mathbf{F}(\boldsymbol{X}(t), t)$ is simply the time-dependent mapping between $\boldsymbol{g}_i(0)$ and $\boldsymbol{g}_i(t)$ for material point $\boldsymbol{X}$. It is called the *deformation gradient tensor*. (For brevity of notation, we occasionally drop the dependence of the $\boldsymbol{g}_i$ on $\boldsymbol{X}$.) Since $\boldsymbol{g}_i(0) = \boldsymbol{e}_i$, we have that

$$\boldsymbol{g}_i(t) = \mathbf{F}(\boldsymbol{X}(t), t) \cdot \boldsymbol{e}_i \tag{1.10}$$

or equivalently, the ith column of $\mathbf{F}(\boldsymbol{X}(t), t)$ is the basis vector $\boldsymbol{g}_i(t)$: $F_{ji}(\boldsymbol{X}(t), t) = g_{ji}(\boldsymbol{X}(t), t)$. Inserting (1.10) into (1.8) and factoring out $\boldsymbol{e}_i$ yields that

$$\frac{d\mathbf{F}}{dt} = \mathbf{L} \cdot \mathbf{F} \tag{1.11}$$

with initial condition $\mathbf{F}(\boldsymbol{X}(0), 0) = \boldsymbol{\delta}$. In the special case of a steady linear flow, where $\mathbf{L}$ is constant, $\mathbf{F} = e^{\mathbf{L}t}$ and the $\boldsymbol{g}_i$ are independent of position.

Because the $\boldsymbol{g}_i$ evolve as material lines, Equation (1.9) also holds with $\boldsymbol{g}_i$ replaced with $\Delta \boldsymbol{X}$:

$$\Delta \boldsymbol{X}(t) = \mathbf{F}(\boldsymbol{X}(t), t) \cdot \Delta \boldsymbol{X}(0). \tag{1.12}$$

Thus the deformation gradient tensor $\mathbf{F}$ contains all information needed to determine the evolution of a material line during a deformation.[5] Recalling that the ith column of $\mathbf{F}(\boldsymbol{X}(t), t)$ is the basis vector $\boldsymbol{g}_i(t)$, we can rewrite (1.12) in the coordinate system defined by these vectors:

$$\Delta \boldsymbol{X}(t) = \Delta X_1(0)\boldsymbol{g}_1(t) + \Delta X_2(0)\boldsymbol{g}_2(t) + \Delta X_3(0)\boldsymbol{g}_3(t). \tag{1.13}$$

For $t > 0$ the basis vectors $\boldsymbol{g}_i(t)$ are not generally orthogonal. However, it is always possible to find a *reciprocal basis* comprised of vectors $\boldsymbol{h}_j(t)$ that satisfy the so-called *biorthogonality* condition $\boldsymbol{g}_i(t) \cdot \boldsymbol{h}_j(t) = \delta_{ij}$. For the parallelepiped whose edges are defined by the $\boldsymbol{g}_i$, the normal vectors to the faces of the parallelepiped are defined by the $\boldsymbol{h}_i$, as shown in Figure 1.5. Now thinking of coordinate planes that are embedded in the material and evolve with it, the $\boldsymbol{h}_i$ are basis vectors that are orthogonal to these planes and are said to be *covariant*. Since the basis vectors $\boldsymbol{g}_i(t)$ are contained in the columns of the deformation gradient tensor $\mathbf{F}(t)$, the columns of $(\mathbf{F}(t)^{\mathrm{T}})^{-1}$ will contain the reciprocal basis vectors $\boldsymbol{h}_j(t)$. These vectors satisfy

$$\frac{d\boldsymbol{h}_j}{dt} = -\mathbf{L}^{\mathrm{T}} \cdot \boldsymbol{h}_j. \tag{1.14}$$

Problem 1.4 illustrates one context in which this reciprocal basis arises.

Just as the vectors $\boldsymbol{g}_i$ comprise a codeforming basis set for representing vectors associated with a deforming material, the dyads $\boldsymbol{g}_i\boldsymbol{g}_j$ form a basis for representing second-order tensors. In index notation, where $g_{kj} = \boldsymbol{e}_k \cdot \boldsymbol{g}_j$, these dyads evolve as follows:

$$\begin{aligned}\left(\frac{d\boldsymbol{g}_i\boldsymbol{g}_j}{dt}\right)_{lk} = \frac{dg_{li}g_{kj}}{dt} &= \frac{dg_{li}}{dt}g_{kj} + g_{li}\frac{dg_{kj}}{dt}\\ &= L_{lm}g_{mi}g_{kj} + g_{li}L_{km}g_{mj}\\ &= L_{lm}g_{mi}g_{kj} + g_{li}g_{mj}L^{\mathrm{T}}_{mk},\end{aligned}$$

which we can rewrite

$$\frac{d\boldsymbol{g}_i\boldsymbol{g}_j}{dt} = \boldsymbol{g}_i\boldsymbol{g}_j \cdot \nabla\boldsymbol{v} + \nabla\boldsymbol{v}^{\mathrm{T}} \cdot \boldsymbol{g}_i\boldsymbol{g}_j. \tag{1.15}$$

[5] In mathematical terms, $\mathbf{F}$ is the *fundamental solution matrix* for (1.2).

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More generally, we can define the *Green tensor*[6]

$$\mathbf{B}(\boldsymbol{X}(t),t) = \sum_{i=1}^{3}\sum_{i=1}^{3} \boldsymbol{g}_i \boldsymbol{g}_j. \tag{1.16}$$

By construction, at $t = 0$, $\mathbf{B} = \boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is the identity tensor. Using (1.15), we can write that

$$\frac{d\mathbf{B}(\boldsymbol{X}(t),t)}{dt} - \Big(\mathbf{B}(\boldsymbol{X}(t),t)\cdot\nabla\boldsymbol{v}(\boldsymbol{X}(t),t) + \nabla\boldsymbol{v}^{\mathrm{T}}(\boldsymbol{X}(t),t)\cdot\mathbf{B}(\boldsymbol{X}(t),t)\Big) = \mathbf{0}. \tag{1.17}$$

If we think about $\mathbf{B}$ as a tensor field, i.e., as a function of position $\boldsymbol{x}$ in the flow field rather than as a tensor attached to a particular material point $\boldsymbol{X}(t)$, then the time derivative is replaced by a substantial derivative[7]

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla,$$

in which case

$$\frac{D\mathbf{B}(\boldsymbol{x},t)}{Dt} - \Big(\mathbf{B}(\boldsymbol{x},t)\cdot\nabla\boldsymbol{v}(\boldsymbol{x},t) + \nabla\boldsymbol{v}^{\mathrm{T}}(\boldsymbol{x},t)\cdot\mathbf{B}(\boldsymbol{x},t)\Big) = \mathbf{0}. \tag{1.18}$$

The quantity on the left-hand side of this expression is called the *contravariant convected derivative* or *upper convected derivative* of $\mathbf{B}$ and is denoted $\mathbf{B}_{(1)}$ or $\overset{\nabla}{\mathbf{B}}$. This derivative is the rate of change of a tensor relative to a coordinate system that is deforming with the material. The Green tensor is fundamentally important to the theory of elasticity and viscoelasticity, in part because the stress tensor for a simple model of a material called the neo-Hookean solid is proportional to it. We will see this object again in Section 8.6, where it arises naturally in a model of the dynamics of dilute polymer solutions. The Green tensor is related to $\mathbf{F}$ by

$$\mathbf{B}(\boldsymbol{X}(t),t) = \mathbf{F}(\boldsymbol{X}(t),t)\cdot\mathbf{F}^{\mathrm{T}}(\boldsymbol{X}(t),t). \tag{1.19}$$

(See Problem 1.2.)

[6] We use the nomenclature recommended by the International Union of Pure and Applied Chemistry (IUPAC) (Kaye et al. 1998). This tensor is also called the *left Cauchy–Green tensor* or sometimes the *Finger tensor*, although according to the IUPAC standard the latter term refers to the quantity $\left(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}\right)^{-1}$. The *right Cauchy–Green tensor* $\mathbf{C}$, also called the *Cauchy tensor* , is given by $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$.

[7] Consider a scalar field $f(\boldsymbol{x}(t),t)$. The rate of change of this field as measured by an observer moving with velocity $d\boldsymbol{x}/dt = \boldsymbol{v}$ (i.e., moving as a material point $\boldsymbol{X}(t)$) is given by the *substantial derivative* Df/Dt. In Cartesian coordinates, we can use the chain rule to write this as

$$\begin{aligned}\frac{Df(\boldsymbol{x},t)}{Dt} &= \frac{\partial f}{\partial t} + \sum_{i=1}^{3}\frac{dx_i}{dt}\frac{\partial f}{\partial x_i}\\ &= \frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i\frac{\partial f}{\partial x_i}\\ &= \frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f.\end{aligned}$$

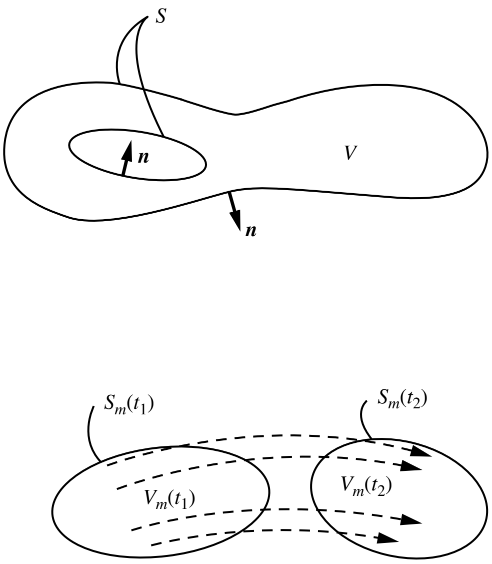

Figure 1.6 Arbitrary stationary (top) and material (bottom) volumes within a material for derivation of conservation laws.

A number of other deformation tensors and convected derivatives also arise in studies of complex fluids and elastic solids. Section 9.4 illustrates the importance of these derivatives in the general context of constitutive models for stress in a material.

1.2 Conservation Equations

1.2.1 Conservation of Mass

Consider an arbitrarily chosen stationary volume V with boundary S within a material of mass density ρ as shown in Figure 1.6. The outward unit normal vector for the volume is denoted $\boldsymbol{n}$. Given that the mass flux at any position $\boldsymbol{x}$ in the volume is $\rho\boldsymbol{v}$, the mass balance for this domain can be written as

$$\frac{d}{dt}\int_V \rho\, dV = -\int_S \boldsymbol{n}\cdot(\rho\boldsymbol{v})\, dS. \tag{1.20}$$

This equation simply states that the rate of accumulation of mass in the domain is equal to the integral over the boundary of the mass flux into the volume $-\boldsymbol{n}\cdot(\rho\boldsymbol{v})$ – mass is neither created nor destroyed within the volume V. Applying the divergence theorem (Equation (A.10) in Appendix A.2) to the right-hand side and rearranging yields that

$$\frac{d}{dt}\int_V \rho\, dV + \int_V \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\, dV = 0. \tag{1.21}$$

Since the volume V is stationary, this can be rewritten

$$\int_V \frac{\partial\rho}{\partial t}\, dV + \int_V \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\, dV = 0$$

or

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) \right) dV = 0.$$

The domain of integration is arbitrary, so the only way the integral can vanish is for the integrand to vanish pointwise:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{1.22}$$

at every point in the domain. This is the statement of conservation of mass at a point in a continuous medium and is known as the *continuity equation*. We will exclusively consider *incompressible flows* during which density changes are negligible and the continuity equation reduces to

$$\nabla \cdot \boldsymbol{v} = 0. \tag{1.23}$$

1.2.2 Conservation of Momentum

To address conservation of momentum, we take a slightly different approach than we did for mass conservation. Consider an arbitrary *material volume* $V_{\mathrm{M}}(t)$, by which we mean a volume whose elements move with the material velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ (Figure 1.6). Therefore points on the boundary of this volume $S_{\mathrm{M}}(t)$ are also moving with this velocity. The total momentum of this volume is

$$\int_{V_{\mathrm{M}}(t)} \rho \boldsymbol{v} \, dV$$

and Newton's second law applied to it becomes

$$\frac{d}{dt} \int_{V_{\mathrm{M}}(t)} \rho \boldsymbol{v} \, dV = \boldsymbol{F}_V + \boldsymbol{F}_S. \tag{1.24}$$

Here the left-hand side is the rate of change of momentum of the volume, $\boldsymbol{F}_V$ incorporates forces exerted on the material in the volume by external fields that act at each point in the material and $\boldsymbol{F}_S$ incorporates forces exerted across the boundary of the volume by the neighboring material – stresses.

Many body forces – gravitational, electrical, magnetic – can act at each point in a material in an external field. For the moment, we do not specify the nature of these forces but simply write that

$$\boldsymbol{F}_V = \int_{V_{\mathrm{M}}(t)} \boldsymbol{f}(\boldsymbol{x}) \, dV, \tag{1.25}$$

where $\boldsymbol{f}(\boldsymbol{x})$ is an arbitrary position-dependent force density.

The net surface force is the integral of the stresses exerted across the boundary of V. Denoting the stress vector (sometimes called the traction vector) exerted at a point on the boundary by the neighboring material outside the boundary as $\boldsymbol{t}(\boldsymbol{x})$, this force can be written

$$\boldsymbol{F}_S = \int_{S_{\mathrm{M}}(t)} \boldsymbol{t}(\boldsymbol{x}) \, dS.$$