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978-1-107-02462-5 - Cox Rings

Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen and Antonio Laface

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