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978-1-107-02462-5 - Cox Rings

Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen and Antonio Laface

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## COX RINGS

Cox rings are significant global invariants of algebraic varieties, naturally generalizing homogeneous coordinate rings of projective spaces. This book provides a largely self-contained introduction to Cox rings, with a particular focus on concrete aspects of the theory. Besides the rigorous presentation of the basic concepts, other central topics include the case of finitely generated Cox rings and its relation to toric geometry; various classes of varieties with group actions; the surface case; and applications in arithmetic problems, in particular Manin's conjecture. The introductory chapters require only basic knowledge of algebraic geometry. The more advanced chapters also touch on algebraic groups, surface theory, and arithmetic geometry.

Each chapter ends with exercises and problems. These comprise mini-tutorials and examples complementing the text, guided exercises for topics not discussed in the text, and, finally, several open problems of varying difficulty.

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# Cox Rings

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