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Introduction

The basic principles regarding Cox rings become visible already in the classical example of the projective space $\mathbb{P}^n$ over a field $\mathbb{K}$, which we assume to be algebraically closed and of characteristic zero. The elements of $\mathbb{P}^n$ are the lines $\ell \subseteq \mathbb{K}^{n+1}$ through the origin $0 \in \mathbb{K}^{n+1}$. Such a line ℓ is concretely specified by its *homogeneous coordinates* $[z_0, \ldots, z_n]$, where $(z_0, \ldots, z_n)$ is *any* point on ℓ, different from the origin. Hence, this description comes with an intrinsic ambiguity. More formally speaking, that means that we should regard the projective space as a quotient by a group action

$$\begin{array}{ccc} \mathbb{K}^{n+1} \setminus \{0\} & \subseteq & \mathbb{K}^{n+1} \\ {\scriptstyle z \mapsto [z]} \big\downarrow {\scriptstyle /\mathbb{K}^*} & & \\ \mathbb{P}^n & & \end{array}$$

where $\mathbb{K}^*$ acts on $\mathbb{K}^{n+1}$ via scalar multiplication. This presentation of the projective space $\mathbb{P}^n$ as the quotient of its *characteristic space* $\mathbb{K}^{n+1} \setminus \{0\}$ by the action of the *characteristic torus* $\mathbb{K}^*$ is the geometric way of thinking of Cox rings. In algebraic terms, the action of $\mathbb{K}^*$ on $\mathbb{K}^{n+1}$ is encoded by the associated decomposition of the polynomial ring into homogeneous parts

$$\mathbb{K}[T_0, \ldots, T_n] = \bigoplus_{k \geq 0} \mathbb{K}[T_0, \ldots, T_n]_k,$$

where $\mathbb{K}[T_0, \ldots, T_n]_k$ is the vector space of homogeneous polynomials f of degree k, which means that $f(tz) = t^k f(z)$ holds for all t and z. The polynomial ring together with this classical grading is the *Cox ring* of the projective space. Note that to construct $\mathbb{P}^n$ as a $\mathbb{K}^*$-quotient, we have to remove the origin, which is the vanishing locus of the *irrelevant ideal*, from the *total coordinate space* $\mathbb{K}^{n+1}$. The Cox ring can be seen in terms of algebraic geometry intrinsically

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from the projective space as

$$\mathbb{K}[T_0, \ldots, T_n] \;=\; \bigoplus_{k \geq 0} \mathbb{K}[T_0, \ldots, T_n]_k \;\cong\; \bigoplus_{k \in \mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{O}(kD)),$$

where the class of the hyperplane $D := V(T_0) \subseteq \mathbb{P}^n$ freely generates the *divisor class group* $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$ and the sections $1, T_1/T_0, \ldots, T_n/T_0$ of degree 1 on the right-hand side correspond to the homogeneous generators $T_0, \ldots, T_n$ on the left-hand side. The brief discussion of this simple example shows us that Cox rings are located in the intersection of three fields: graded algebras, group actions and quotients, and divisors and their section rings.

Let us also take a brief look at the arithmetic aspects. The rational points $[z_0, \ldots, z_n]$ in the projective space $\mathbb{P}^n$ over $\mathbb{Q}$ are parameterized uniquely up to sign by primitive vectors, that is, tuples of coprime integers $z_0, \ldots, z_n$. This description of rational points is related to Cox rings via the diagram

$$\begin{array}{ccc} \mathbb{Z}^{n+1}_{\mathrm{prim}} \setminus \{0\} & \subseteq & \mathbb{Z}^{n+1} \\ {\scriptstyle z \mapsto [z]} \Big\downarrow {\scriptstyle /\mathbb{Z}^*} & & \\ \mathbb{P}^n(\mathbb{Q}) & & \end{array}$$

A typical problem is to estimate the number of rational points with bounded height $H([z_0, \ldots, z_n]) := \max\{|z_0|, \ldots, |z_n|\}$, which in our example essentially amounts to estimating the number of lattice points in an $(n+1)$-dimensional box. This is an instance of Manin's conjecture on the number of rational points of bounded height on Fano varieties.

The current interest in Cox rings has several sources. A first one dates back to the 1970s when Colliot-Thélène and Sansuc [96, 98] introduced universal torsors as a tool in arithmetic geometry in particular to investigate the existence of rational points on varieties. In the last few years, Salberger's approach [263] to study the distribution of rational points via universal torsors and their explicit representations in terms of Cox rings caused a considerable surge of research. Another source is the occurrence of characteristic spaces and Cox rings in toric geometry in the mid-1990s in work of Audin [23], Cox [104] and others, which had a tremendous impact on this field. Five years later, Hu and Keel [176] observed the fundamental connection between Mori theory and geometric invariant theory via Cox rings; one of the main insights is that, roughly speaking, finite generation of the Cox ring is equivalent to an optimal behavior with respect to the minimal model program. This put the toric case into a much more general framework and established Cox rings as an active field of research in algebraic geometry. For example, the explicit presentation of the Cox ring of a given variety in terms of generators and relations is a central question. The research in this direction was initiated by the work of Batyrev/Popov [33] and Hassett/Tschinkel [162] on (weak) del Pezzo surfaces.

The intention of this book is to provide an elementary access to Cox rings and their applications in algebraic and arithmetic geometry with a particular focus on the new, concrete aspects that Cox rings bring into these fields. The introductory part, consisting of the first three chapters, requires basic knowledge in algebraic geometry, and, in addition, some familiarity with toric varieties is helpful. The subsequent three chapters consider also more advanced topics such as algebraic groups, surface theory, and arithmetic questions.

Chapter 1 provides the mathematical framework for the ideas occurring in the preceding example discussion. We present the basics on graded algebras, quasitorus actions and their quotients, and divisors and sheaves of divisorial algebras. Building on this, we define an essentially unique Cox ring for any irreducible, normal variety with only constant invertible functions and a finitely generated divisor class group:

$$\mathcal{R}(X) := \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

First results concern the algebraic properties of the Cox ring, in particular the divisibility properties: Cox rings are factorially graded rings in the sense that we have unique factorization for homogeneous elements. The further main results of the chapter elaborate the relations between the Cox ring and its geometric counterpart, the presentation of the underlying variety as a quotient of the characteristic space by the action of the characteristic quasitorus. For smooth varieties, this quotient presentation equals the universal torsor and in general it dominates the universal torsor.

Chapter 2 discusses the concepts provided in Chapter 1 for the example class of toric varieties; these come with an action of an algebraic torus having a dense open orbit. The basic feature of toric varieties is their complete description in terms of combinatorial data, so-called lattice fans. Approaching toric varieties via quotient presentations turns out to be combinatorial as well, and the describing data, which we call lattice bunches, correspond to lattice fans via linear Gale duality. Besides being illustrative examples, toric varieties are important in subsequent chapters as adapted ambient varieties.

Chapter 3 is devoted to varieties with a finitely generated Cox ring. In this setting, the varieties sharing the same Cox ring all occur as quotients of open subsets of their common total coordinate space, the spectrum of the Cox ring. Geometric invariant theory gives us a concrete combinatorial description of the possible characteristic spaces, which finally leads to the encoding of our varieties by “bunched rings.” The resulting picture shares many combinatorial features with toric geometry. In fact, the varieties inherit many properties from a canonical toric ambient variety. We take a look from the combinatorial aspect to invariants such as Picard groups; local divisor class groups; and the cones of effective, movable, semiample, and ample divisors. Moreover, we treat singularities, intersection numbers, and the Mori chamber structure of the effective cone. A particularly interesting class are the rational varieties with a torus action of complexity 1, for example, $\mathbb{K}^*$-surfaces. Here, the bunched ring description leads to a very efficient approach to the geometry; for example, one obtains a concrete combinatorial resolution of singularities.

Chapter 4 begins with a study of Cox rings of embedded varieties and the effect of modifications, for example, blow-ups on the Cox ring. Then we investigate the various quotient presentations of a variety and show that they are all dominated by the characteristic space, playing here a similar role as the universal covering in topology. The problem of lifting group actions to quotient presentations and the automorphism group are investigated. We provide various criteria for finite generation of the Cox ring, for example, Knop's criterion for unirational varieties with a group action of complexity 1, a characterization via the multiplication map, and the characterization in terms of Mori theory due to Hu and Keel. Moreover, we relate Cox rings of blow-ups of the projective space to invariant rings of unipotent group actions following Nagata. For varieties coming with a torus action, we describe the Cox ring in terms of isotropy groups and a certain quotient; this generalizes the toric case and gives the foundation for the bunched ring approach to the more general case of complexity 1. Finally, we take a look at almost homogeneous varieties. After describing the Cox ring of a homogeneous space, we discuss embeddings with a small boundary in terms of bunched rings and then turn to Brion's description of Cox rings of spherical varieties and wonderful compactifications.

In Chapter 5, we take a close look at Cox rings of complex algebraic surfaces. A first general part is devoted to the classification of smooth Mori dream surfaces. We present a complete picture for surfaces with nonnegative anticanonical Iitaka dimension, and study in detail the cases of elliptic rational surfaces, K3 surfaces, and Enriques surfaces. Then we turn to the explicit description of Cox rings by generators and relations. For del Pezzo surfaces, we show that the Cox ring is generated in anticanonical degree 1 and that the ideal of relations is generated by quadrics. A discussion of the relations between Cox rings of del Pezzo surfaces and flag varieties ends this part. Then we return to K3 surfaces. Here we provide a detailed study in the case of Picard number 2 and complete results are obtained for double covers of del Pezzo surfaces and of blow-ups of Hirzebruch surfaces in at most three points. Finally, we develop the theory of rational $\mathbb{K}^*$-surfaces. Here we allow singularities and show how their minimal resolution is encoded in the Cox ring. As an example class, we present the Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces in terms of their Cox rings.

The aim of Chapter 6 is to indicate how Cox rings and universal torsors can be applied to arithmetic questions regarding rational points on varieties. We begin by discussing Colliot-Thélène's and Sansuc's theory of universal torsors over not necessarily algebraically closed fields, and explore the connection to characteristic spaces and Cox rings. Then we enter the problem of the existence of rational points on varieties over number fields. We discuss the Hasse principle and weak approximation. The failure of these principles is often explained by Brauer–Manin obstructions, and we indicate how they can be approached via universal torsors and give an overview of the existing results. Then we turn to Manin's conjecture. For del Pezzo surfaces, it is known in many cases, and a general strategy emerges. We discuss this strategy in detail and show how it can be applied to prove Manin's conjecture for a singular cubic surface.

Each chapter is followed by a choice of exercises and problems. The collections comprise small general background tutorials and examples complementing the text as well as guided exercises to topics going beyond the text including references and, finally, we pose several open problems (*) of varying presumed difficulty.

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1

Basic concepts

In this chapter we introduce the Cox ring and, more generally, the *Cox sheaf* and its geometric counterpart, the *characteristic space*. In addition, algebraic and geometric aspects are discussed. Section 1.1 is devoted to commutative algebras graded by monoids. In Section 1.2, we recall the correspondence between actions of quasitori (also called diagonalizable groups) on affine varieties and affine algebras graded by abelian groups and provide the necessary background on good quotients. Section 1.3 is a first step toward constructing Cox rings. Given an irreducible, normal variety X and a finitely generated subgroup $K \subseteq \mathrm{WDiv}(X)$ of the group of Weil divisors, we consider the associated *sheaf of divisorial algebras*

$$\mathcal{S} = \bigoplus_{D \in K} \mathcal{O}_X(D).$$

We present criteria for local finite generation and consider the relative spectrum. A first result says that $\Gamma(X, \mathcal{S})$ is a unique factorization domain if K generates the divisor class group $\mathrm{Cl}(X)$. Moreover, we characterize divisibility in the ring $\Gamma(X, \mathcal{S})$ in terms of divisors on X. In Section 1.4, the Cox sheaf of an irreducible, normal variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ is introduced; roughly speaking it is given as

$$\mathcal{R} = \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{O}_X(D).$$

The Cox ring then is the corresponding ring of global sections. In the case of a free divisor class group well-definedness is straightforward. The case of torsion needs some effort; the precise way to define $\mathcal{R}$ then is to take the quotient of an appropriate sheaf of divisorial algebras with respect to a certain ideal sheaf. Basic algebraic properties and divisibility theory of the Cox ring are investigated in Section 1.5. Finally, in Section 1.6, we study the characteristic space, that is, the relative spectrum $\widehat{X} = \mathrm{Spec}_X \mathcal{R}$ of the Cox sheaf. It comes with an action of the *characteristic quasitorus* $H = \operatorname{Spec} \mathbb{K}[\operatorname{Cl}(X)]$ and a good quotient $\widehat{X} \to X$. We relate geometric properties of X to properties of this action and describe the characteristic space in terms of geometric invariant theory.

1.1 Graded algebras

1.1.1 Monoid graded algebras

We recall basic notions on algebras graded by abelian monoids. In this subsection, R denotes a commutative ring with a unit element.

Definition 1.1.1.1 Let K be an abelian monoid. A *K-graded R-algebra* is an associative, commutative R-algebra A with a unit and a direct sum decomposition

$$A = \bigoplus_{w \in K} A_w$$

into R-submodules $A_w \subseteq A$ such that $A_w \cdot A_{w'} \subseteq A_{w+w'}$ holds for any two elements $w, w' \in K$. The R-submodules $A_w \subseteq A$ are the *(K-)homogeneous components* of A. An element $f \in A$ is *(K-)homogeneous* if $f \in A_w$ holds for some $w \in K$, and in this case w is called the *degree* of f. We write $A_\times \subseteq A$ for the multiplicative monoid of homogeneous elements.

We also speak of a K-graded R-algebra as a monoid graded algebra or just as a graded algebra. To compare R-algebras A and A' that are graded by different abelian monoids K and K', we work with the following notion of a morphism.

Definition 1.1.1.2 A *morphism* from a K-graded algebra A to a K'-graded algebra A' is a pair $(\psi, \widetilde{\psi})$, where $\psi\colon A \to A'$ is a homomorphism of R-algebras, $\widetilde{\psi}\colon K \to K'$ is a homomorphism of abelian monoids, and

$$\psi(A_w) \subseteq A'_{\widetilde{\psi}(w)}$$

holds for every $w \in K$. In the case $K = K'$ and $\widetilde{\psi} = \mathrm{id}_K$, we denote a morphism of graded algebras just by $\psi\colon A \to A'$ and also refer to it as a *(K-)graded homomorphism*.

Example 1.1.1.3 Given an abelian monoid K and $w_1, \ldots, w_r \in K$, the polynomial ring $R[T_1, \ldots, T_r]$ can be turned into a K-graded R-algebra by setting

$$R[T_1, \ldots, T_r]_w := \left\{ \sum_{\nu \in \mathbb{Z}^r_{\geq 0}} a_\nu T^\nu;\ a_\nu \in R,\ \nu_1 w_1 + \cdots + \nu_r w_r = w \right\}.$$

This K-grading is determined by $\deg(T_i) = w_i$ for $1 \le i \le r$. Moreover, $R[T_1, \ldots, T_r]$ comes with the natural $\mathbb{Z}^r_{\geq 0}$-grading given by

$$R[T_1, \ldots, T_r]_\nu := R \cdot T^\nu,$$

and we have a canonical morphism $(\psi, \widetilde{\psi})$ from $R[T_1, \ldots, T_r]$ to itself, where $\psi = \mathrm{id}$ and $\widetilde{\psi} \colon \mathbb{Z}^r_{\geq 0} \to K$ sends ν to $\nu_1 w_1 + \cdots + \nu_r w_r$.

For any abelian monoid K, we denote by $K^\pm$ the associated group of differences and by $K_\mathbb{Q} := K^\pm \otimes_\mathbb{Z} \mathbb{Q}$ the associated rational vector space. Note that we have canonical maps $K \to K^\pm \to K_\mathbb{Q}$, where the first one is injective if and only if K admits cancellation and the second one is injective if and only if $K^\pm$ is torsion free. Given $w \in K$, we allow ourselves to write $w \in K^\pm$ and $w \in K_\mathbb{Q}$ for the respective images.

Definition 1.1.1.4 Let A be a K-graded R-algebra. The *weight monoid* of A is the submonoid $S(A) \subseteq K$ generated by all $w \in K$ with $A_w \neq 0$. The *weight group* of A is the subgroup $K(A) \subseteq K^\pm$ generated by $S(A) \subseteq K$. The *weight cone* of A is the convex cone $\omega(A) \subseteq K_\mathbb{Q}$ generated by $S(A) \subseteq K$.

By an *integral R-algebra*, we mean an R-algebra $A \neq 0$ without zero divisors. Note that for an integral R-algebra A graded by an abelian monoid K, the weight monoid of A is given as

$$S(A) \;=\; \{w \in K;\; A_w \neq 0\} \;\subseteq\; K.$$

We recall the construction of the algebra associated with an abelian monoid; it defines a covariant functor from the category of abelian monoids to the category of monoid graded algebras.

Construction 1.1.1.5 Let K be an abelian monoid. As an R-module, the associated *monoid algebra* over R is given by

$$R[K] := \bigoplus_{w \in K} R \cdot \chi^w$$

and its multiplication is defined by $\chi^w \cdot \chi^{w'} := \chi^{w+w'}$. If K' is a further abelian monoid and $\widetilde{\psi} \colon K \to K'$ is a homomorphism, then we have a homomorphism

$$\psi := R[\widetilde{\psi}] \colon R[K] \;\to\; R[K'], \qquad \chi^w \;\mapsto\; \chi^{\widetilde{\psi}(w)}.$$

The pair $(\psi, \widetilde{\psi})$ is a morphism from the K-graded algebra $R[K]$ to the K'-graded algebra $R[K']$, and this assignment is functorial.

Note that the monoid algebra $R[K]$ has K as its weight monoid, and $R[K]$ is finitely generated over R if and only if the monoid K is finitely generated. In general, if a K-graded algebra A is finitely generated over R, then its weight monoid is finitely generated and its weight cone is *polyhedral*, that is, the set of nonnegative linear combinations over a given finite collection of vectors.

Construction 1.1.1.6 (Trivial extension) Let $K \subseteq K'$ be an inclusion of abelian monoids and A a K-graded R-algebra. Then we obtain an K'-graded R-algebra A' by setting

$$A' := \bigoplus_{u \in K'} A'_u, \qquad A'_u := \begin{cases} A_u & \text{if } u \in K, \\ \{0\} & \text{else.} \end{cases}$$

Construction 1.1.1.7 (Lifting) Let $G\colon \widetilde{K} \to K$ be a homomorphism of abelian monoids and A a K-graded R-algebra. Then we obtain a $\widetilde{K}$-graded R-algebra

$$\widetilde{A} := \bigoplus_{u \in \widetilde{K}} \widetilde{A}_u, \qquad \widetilde{A}_u := A_{G(u)}.$$

Definition 1.1.1.8 Let A be a K-graded R-algebra. An ideal $I \subseteq A$ is called *(K-)homogeneous* if it is generated by (K-)homogeneous elements.

An ideal $I \subseteq A$ of a K-graded R-algebra A is homogeneous if and only if it has a direct sum decomposition

$$I = \bigoplus_{w \in K} I_w, \qquad I_w := I \cap A_w.$$

Construction 1.1.1.9 (Graded factor algebra) Let A be a K-graded R-algebra and $I \subseteq A$ a homogeneous ideal. Then the factor algebra A/I is K-graded by

$$A/I = \bigoplus_{w \in K} (A/I)_w \qquad (A/I)_w := A_w + I.$$

Moreover, for each homogeneous component $(A/I)_w \subseteq A/I$, one has a canonical isomorphism of R-modules

$$A_w/I_w \to (A/I)_w, \qquad f + I_w \mapsto f + I.$$

Construction 1.1.1.10 Let A be a K-graded R-algebra, and $\widetilde{\psi}\colon K \to K'$ be a homomorphism of abelian monoids. Then one may consider A as a K'-graded algebra with respect to the *coarsened grading*

$$A = \bigoplus_{u \in K'} A_u, \qquad A_u := \bigoplus_{\widetilde{\psi}(w) = u} A_w.$$

Example 1.1.1.11 Let $K = \mathbb{Z}^2$ and consider the K-grading of $R[T_1, \ldots, T_5]$ given by $\deg(T_i) = w_i$, where

$$w_1 = (-1, 2), \quad w_2 = (1, 0), \quad w_3 = (0, 1), \quad w_4 = (2, -1), \quad w_5 = (-2, 3).$$

Then the polynomial $T_1T_2 + T_3^2 + T_4T_5$ is K-homogeneous of degree $(0, 2)$, and thus we have a K-graded factor algebra

$$A = R[T_1, \dots, T_5]/\langle T_1T_2 + T_3^2 + T_4T_5\rangle.$$

The standard $\mathbb{Z}$-grading of the algebra A with $\deg(T_1) = \dots = \deg(T_5) = 1$ may be obtained by coarsening via the homomorphism $\widetilde{\psi}\colon \mathbb{Z}^2 \to \mathbb{Z}$, $(a, b) \mapsto a + b$.

Proposition 1.1.1.12 *Let A be a $\mathbb{Z}^r$-graded R-algebra satisfying $ff' \neq 0$ for any two nonzero homogeneous $f, f' \in A$. Then the following statements hold.*

(i) *The algebra A is integral.*
(ii) *If gg' is homogeneous for $0 \neq g, g' \in A$, then g and g' are homogeneous.*
(iii) *Every unit $f \in A^*$ is homogeneous.*

Proof Fix a lexicographic ordering on $\mathbb{Z}^r$. Given two nonzero $g, g' \in A$, write $g = \sum f_u$ and $g' = \sum f'_u$ with homogeneous f_u and f'_u. Then the maximal (minimal) component of gg' is $f_w f'_{w'} \neq 0$, where f_w and $f'_{w'}$ are the maximal (minimal) components of f and f' respectively. The first two assertions follow. For the third one observe that $1 \in A$ is homogeneous (of degree zero). □

1.1.2 Veronese subalgebras

We introduce Veronese subalgebras of monoid graded algebras and present statements relating finite generation of the algebra to finite generation of a given Veronese subalgebra and vice versa.

We begin with basic observations on finite generation of monoids. The first one is a generalization of the classical Gordan lemma which asserts that for any convex polyhedral cone $\sigma \subseteq \mathbb{Q}^r$, the monoid $\sigma \cap \mathbb{Z}^r$ is finitely generated.

Proposition 1.1.2.1 *Let K be a finitely generated abelian group and $L \subseteq M \subseteq K$ submonoids. If L is finitely generated and every $w \in M$ admits an $n \in \mathbb{Z}_{\geq 1}$ with $nw \in L$, then M is finitely generated.*

Proof First we prove Gordan's lemma. Consider $K = \mathbb{Z}^r$; let $L \subseteq K$ be generated by $w_1, \dots, w_s \in \mathbb{Z}^r$ and $M = \sigma \cap \mathbb{Z}^r$ the monoid of integral points inside the convex cone $\sigma \subseteq \mathbb{Q}^r$ generated by $w_1, \dots, w_s$. Then M is generated by the finite subset

$$([0, 1] \cdot w_1 + \dots + [0, 1] \cdot w_s) \cap \mathbb{Z}^r \subseteq M.$$

We turn to the general case. Choose an epimorphism $\alpha\colon \mathbb{Z}^r \to K$. Then also $K' := \mathbb{Z}^r$ with $L' := \alpha^{-1}(L)$ and $M' := \alpha^{-1}(M)$ satisfy the assumptions. So, it suffices to show that M' is finitely generated. Take generators $w_1, \dots, w_s \in \mathbb{Z}^r$