Regression and Other Stories

Many textbooks on regression focus on theory and the simplest of examples. Real statistical problems, however, are complex and subtle. This is not a book about the theory of regression. It is a book about how to use regression to solve real problems of comparison, estimation, prediction, and causal inference. It focuses on practical issues such as sample size and missing data and a wide range of goals and techniques. It jumps right in to methods and computer code you can use fresh out of the box.

Key features:

- Real examples, real stories from the authors' real-world experience demonstrate what can be achieved by regression and what the limitations are
- Uses computation with the popular open-source programs R and Stan instead of deriving formulas, with all code available online
- Emphasis on using graphics and presentation to understand and check models that have been fit to data
- Practical advice for understanding assumptions and implementing methods for experiments and observational studies
- Smooth transition to logistic regression and generalized linear models
- Clear presentation of key ideas in data collection, sampling, generalization, and causal inference

The authors are experienced researchers who have published articles in hundreds of different scientific journals in fields including statistics, computer science, policy, public health, political science, economics, sociology, and engineering. They have also published articles in the Washington Post, New York Times, Slate, and other public venues. Their previous books include Bayesian Data Analysis, Teaching Statistics: A Bag of Tricks, and Data Analysis and Regression Using Multilevel/Hierarchical Models.

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Regression and Other Stories

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## Preface
What you should be able to do after reading and working through this book xi
Fun chapter titles xii
Additional material for teaching and learning xiii

### Part 1: Fundamentals

#### 1 Overview
1.1 The three challenges of statistics 3
1.2 Why learn regression? 4
1.3 Some examples of regression 5
1.4 Challenges in building, understanding, and interpreting regressions 9
1.5 Classical and Bayesian inference 13
1.6 Computing least squares and Bayesian regression 16
1.7 Bibliographic note 17
1.8 Exercises 17

#### 2 Data and measurement
2.1 Examining where data come from 21
2.2 Validity and reliability 23
2.3 All graphs are comparisons 25
2.4 Data and adjustment: trends in mortality rates 31
2.5 Bibliographic note 33
2.6 Exercises 34

#### 3 Some basic methods in mathematics and probability
3.1 Weighted averages 35
3.2 Vectors and matrices 36
3.3 Graphing a line 37
3.4 Exponential and power-law growth and decline; logarithmic and log-log relationships 38
3.5 Probability distributions 40
3.6 Probability modeling 45
3.7 Bibliographic note 47
3.8 Exercises 47

#### 4 Statistical inference
4.1 Sampling distributions and generative models 49
4.2 Estimates, standard errors, and confidence intervals 50
4.3 Bias and unmodeled uncertainty 55
4.4 Statistical significance, hypothesis testing, and statistical errors 57
4.5 Problems with the concept of statistical significance 60
4.6 Example of hypothesis testing: 55,000 residents need your help! 63
4.7 Moving beyond hypothesis testing 66
CONTENTS

4.8 Bibliographic note 67
4.9 Exercises 67

5 Simulation 69
  5.1 Simulation of discrete probability models 69
  5.2 Simulation of continuous and mixed discrete/continuous models 71
  5.3 Summarizing a set of simulations using median and median absolute deviation 73
  5.4 Bootstrapping to simulate a sampling distribution 73
  5.5 Fake-data simulation as a way of life 76
  5.6 Bibliographic note 76
  5.7 Exercises 76

Part 2: Linear regression 79

6 Background on regression modeling 81
  6.1 Regression models 81
  6.2 Fitting a simple regression to fake data 82
  6.3 Interpret coefficients as comparisons, not effects 84
  6.4 Historical origins of regression 85
  6.5 The paradox of regression to the mean 87
  6.6 Bibliographic note 90
  6.7 Exercises 91

7 Linear regression with a single predictor 93
  7.1 Example: predicting presidential vote share from the economy 93
  7.2 Checking the model-fitting procedure using fake-data simulation 97
  7.3 Formulating comparisons as regression models 99
  7.4 Bibliographic note 101
  7.5 Exercises 101

8 Fitting regression models 103
  8.1 Least squares, maximum likelihood, and Bayesian inference 103
  8.2 Influence of individual points in a fitted regression 107
  8.3 Least squares slope as a weighted average of slopes of pairs 108
  8.4 Comparing two fitting functions: lm and stan_glm 109
  8.5 Bibliographic note 111
  8.6 Exercises 111

9 Prediction and Bayesian inference 113
  9.1 Propagating uncertainty in inference using posterior simulations 113
  9.2 Prediction and uncertainty: predict, posterior_linpred, and posterior_predict 115
  9.3 Prior information and Bayesian synthesis 119
  9.4 Example of Bayesian inference: beauty and sex ratio 121
  9.5 Uniform, weakly informative, and informative priors in regression 123
  9.6 Bibliographic note 128
  9.7 Exercises 128

10 Linear regression with multiple predictors 131
  10.1 Adding predictors to a model 131
  10.2 Interpreting regression coefficients 133
  10.3 Interactions 134
  10.4 Indicator variables 136
  10.5 Formulating paired or blocked designs as a regression problem 139
## CONTENTS

### 10.6 Example: uncertainty in predicting congressional elections  
10.7 Mathematical notation and statistical inference  
10.8 Weighted regression  
10.9 Fitting the same model to many datasets  
10.10 Bibliographic note  
10.11 Exercises  

### 11 Assumptions, diagnostics, and model evaluation  
11.1 Assumptions of regression analysis  
11.2 Plotting the data and fitted model  
11.3 Residual plots  
11.4 Comparing data to replications from a fitted model  
11.5 Example: predictive simulation to check the fit of a time-series model  
11.6 Residual standard deviation $\sigma$ and explained variance $R^2$  
11.7 External validation: checking fitted model on new data  
11.8 Cross validation  
11.9 Bibliographic note  
11.10 Exercises  

### 12 Transformations and regression  
12.1 Linear transformations  
12.2 Centering and standardizing for models with interactions  
12.3 Correlation and “regression to the mean”  
12.4 Logarithmic transformations  
12.5 Other transformations  
12.6 Building and comparing regression models for prediction  
12.7 Models for regression coefficients  
12.8 Bibliographic note  
12.9 Exercises  

### Part 3: Generalized linear models  
13 Logistic regression  
13.1 Logistic regression with a single predictor  
13.2 Interpreting logistic regression coefficients and the divide-by-4 rule  
13.3 Predictions and comparisons  
13.4 Latent-data formulation  
13.5 Maximum likelihood and Bayesian inference for logistic regression  
13.6 Cross validation and log score for logistic regression  
13.7 Building a logistic regression model: wells in Bangladesh  
13.8 Bibliographic note  
13.9 Exercises  

### 14 Working with logistic regression  
14.1 Graphing logistic regression and binary data  
14.2 Logistic regression with interactions  
14.3 Predictive simulation  
14.4 Average predictive comparisons on the probability scale  
14.5 Residuals for discrete-data regression  
14.6 Identification and separation  
14.7 Bibliographic note  
14.8 Exercises
### 15 Other generalized linear models 263
15.1 Definition and notation 263
15.2 Poisson and negative binomial regression 264
15.3 Logistic-binomial model 270
15.4 Probit regression: normally distributed latent data 272
15.5 Ordered and unordered categorical regression 273
15.6 Robust regression using the t model 278
15.7 Constructive choice models 279
15.8 Going beyond generalized linear models 283
15.9 Bibliographic note 286
15.10 Exercises 286

### Part 4: Before and after fitting a regression 289
16 Design and sample size decisions 291
16.1 The problem with statistical power 291
16.2 General principles of design, as illustrated by estimates of proportions 293
16.3 Sample size and design calculations for continuous outcomes 297
16.4 Interactions are harder to estimate than main effects 301
16.5 Design calculations after the data have been collected 304
16.6 Design analysis using fake-data simulation 306
16.7 Bibliographic note 310
16.8 Exercises 310

17 Poststratification and missing-data imputation 313
17.1 Poststratification: using regression to generalize to a new population 313
17.2 Fake-data simulation for regression and poststratification 320
17.3 Models for missingness 322
17.4 Simple approaches for handling missing data 324
17.5 Understanding multiple imputation 326
17.6 Nonignorable missing-data models 332
17.7 Bibliographic note 333
17.8 Exercises 333

### Part 5: Causal inference 337
18 Causal inference and randomized experiments 339
18.1 Basics of causal inference 339
18.2 Average causal effects 342
18.3 Randomized experiments 345
18.4 Sampling distributions, randomization distributions, and bias in estimation 346
18.5 Using additional information in experimental design 347
18.6 Properties, assumptions, and limitations of randomized experiments 350
18.7 Bibliographic note 355
18.8 Exercises 356

19 Causal inference using regression on the treatment variable 363
19.1 Pre-treatment covariates, treatments, and potential outcomes 363
19.2 Example: the effect of showing children an educational television show 364
19.3 Including pre-treatment predictors 367
19.4 Varying treatment effects, interactions, and poststratification 370
19.5 Challenges of interpreting regression coefficients as treatment effects 373
19.6 Do not adjust for post-treatment variables 374
CONTENTS

IX

19.7 Intermediate outcomes and causal paths 376
19.8 Bibliographic note 379
19.9 Exercises 380

20 Observational studies with all confounders assumed to be measured 383
20.1 The challenge of causal inference 383
20.2 Using regression to estimate a causal effect from observational data 386
20.3 Assumption of ignorable treatment assignment in an observational study 388
20.4 Imbalance and lack of complete overlap 391
20.5 Example: evaluating a child care program 394
20.6 Subclassification and average treatment effects 397
20.7 Propensity score matching for the child care example 399
20.8 Restructuring to create balanced treatment and control groups 405
20.9 Additional considerations with observational studies 413
20.10 Bibliographic note 416
20.11 Exercises 417

21 Additional topics in causal inference 421
21.1 Estimating causal effects indirectly using instrumental variables 421
21.2 Instrumental variables in a regression framework 427
21.3 Regression discontinuity: known assignment mechanism but no overlap 432
21.4 Identification using variation within or between groups 440
21.5 Causes of effects and effects of causes 445
21.6 Bibliographic note 449
21.7 Exercises 450

Part 6: What comes next? 455

22 Advanced regression and multilevel models 457
22.1 Expressing the models so far in a common framework 457
22.2 Incomplete data 458
22.3 Correlated errors and multivariate models 459
22.4 Regularization for models with many predictors 459
22.5 Multilevel or hierarchical models 460
22.6 Nonlinear models, a demonstration using Stan 460
22.7 Nonparametric regression and machine learning 464
22.8 Computational efficiency 467
22.9 Bibliographic note 471
22.10 Exercises 471

Appendixes 473

A Computing in R 475
A.1 Downloading and installing R and Stan 475
A.2 Accessing data and code for the examples in the book 476
A.3 The basics 476
A.4 Reading, writing, and looking at data 481
A.5 Making graphs 482
A.6 Working with messy data 484
A.7 Some R programming 488
A.8 Working with rstanarm fit objects 490
A.9 Bibliographic note 492
B 10 quick tips to improve your regression modeling

B.1 Think about variation and replication
B.2 Forget about statistical significance
B.3 Graph the relevant and not the irrelevant
B.4 Interpret regression coefficients as comparisons
B.5 Understand statistical methods using fake-data simulation
B.6 Fit many models
B.7 Set up a computational workflow
B.8 Use transformations
B.9 Do causal inference in a targeted way, not as a byproduct of a large regression
B.10 Learn methods through live examples

References

Author Index

Subject Index
Preface

Existing textbooks on regression typically have some mix of cookbook instruction and mathematical derivation. We wrote this book because we saw a new way forward, focusing on understanding regression models, applying them to real problems, and using simulations with fake data to understand how the models are fit. After reading this book and working through the exercises, you should be able to simulate regression models on the computer and build, critically evaluate, and use them for applied problems.

The other special feature of our book, in addition to its wide range of examples and its focus on computer simulation, is its broad coverage, including the basics of statistics and measurement, linear regression, multiple regression, Bayesian inference, logistic regression and generalized linear models, extrapolation from sample to population, and causal inference. Linear regression is the starting point, but it does not make sense to stop there: once you have the basic idea of statistical prediction, it can be best understood by applying it in many different ways and in many different contexts.

After completing Part 1 of this book, you should have access to the tools of mathematics, statistics, and computing that will allow you to work with regression models. These early chapters should serve as a bridge from the methods and ideas you may have learned in an introductory statistics course. Goals for Part 1 include displaying and exploring data, computing and graphing linear relations, understanding basic probability distributions and statistical inferences, and simulation of random processes to represent inferential and forecast uncertainty.

After completing Part 2, you should be able to build, fit, understand, use, and assess the fit of linear regression models. The chapters in this part of the book develop relevant statistical and computational tools in the context of several applied and simulated-data examples. After completing Part 3, you should be able to similarly work with logistic regression and other generalized linear models. Part 4 covers data collection and extrapolation from sample to population, and in Part 5 we cover causal inference, starting with basic methods using regression for controlled experiments and then considering more complicated approaches adjusting for imbalances in observational data or capitalizing on natural experiments. Part 6 introduces more advanced regression models, and the appendixes include some quick tips and an overview on software for model fitting.

What you should be able to do after reading and working through this book

This text is structured through models and examples, with the intention that after each chapter you should have certain skills in fitting, understanding, and displaying models:

• Part 1: Review key tools and concepts in mathematics, statistics, and computing.
  – Chapter 1: Have a sense of the goals and challenges of regression.
  – Chapter 2: Explore data and be aware of issues of measurement and adjustment.
  – Chapter 3: Graph a straight line and know some basic mathematical tools and probability distributions.
  – Chapter 4: Understand statistical estimation and uncertainty assessment, along with the problems of hypothesis testing in applied statistics.
  – Chapter 5: Simulate probability models and uncertainty about inferences and predictions.
Part 2: Build linear regression models, use them in real problems, and evaluate their assumptions and fit to data.

- Chapter 6: Distinguish between descriptive and causal interpretations of regression, understanding these in historical context.
- Chapter 7: Understand and work with simple linear regression with one predictor.
- Chapter 8: Gain a conceptual understanding of least squares fitting and be able to perform these fits on the computer.
- Chapter 9: Perform and understand probabilistic prediction and simple Bayesian information aggregation, and be introduced to prior distributions and Bayesian inference.
- Chapter 10: Build, fit, and understand linear models with multiple predictors.
- Chapter 11: Understand the relative importance of different assumptions of regression models and be able to check models and evaluate their fit to data.
- Chapter 12: Apply linear regression more effectively by transforming and combining predictors.

Part 3: Build and work with logistic regression and generalized linear models.

- Chapter 13: Fit, understand, and display logistic regression models for binary data.
- Chapter 14: Build, understand, and evaluate logistic regressions with interactions and other complexities.
- Chapter 15: Fit, understand, and display generalized linear models, including the Poisson and negative binomial regression, ordered logistic regression, and other models.

Part 4: Design studies and use data more effectively in applied settings.

- Chapter 16: Use probability theory and simulation to guide data-collection decisions, without falling into the trap of demanding unrealistic levels of certainty.
- Chapter 17: Use poststratification to generalize from sample to population, and use regression models to impute missing data.

Part 5: Implement and understand basic statistical designs and analyses for causal inference.

- Chapter 18: Understand assumptions underlying causal inference with a focus on randomized experiments.
- Chapter 19: Perform causal inference in simple settings using regressions to estimate treatment effects and interactions.
- Chapter 20: Understand the challenges of causal inference from observational data and statistical tools for adjusting for differences between treatment and control groups.
- Chapter 21: Understand the assumptions underlying more advanced methods that use auxiliary variables or particular data structures to identify causal effects, and be able to fit these models to data.

Part 6: Become aware of more advanced regression models.

- Chapter 22: Get a sense of the directions in which linear and generalized linear models can be extended to attack various classes of applied problems.

Appendixes:

- Appendix A: Get started in the statistical software R, with a focus on data manipulation, statistical graphics, and fitting and using regressions.
- Appendix B: Become aware of some important ideas in regression workflow.

After working through the book, you should be able to fit, graph, understand, and evaluate linear and generalized linear models and use these model fits to make predictions and inferences about quantities of interest, including causal effects of treatments and exposures.
PREFACE

Fun chapter titles
The chapter titles in the book are descriptive. Here are more dramatic titles intended to evoke some of the surprise you should feel when working through this material:

• Part 1:
  – Chapter 1: Prediction as a unifying theme in statistics and causal inference.
  – Chapter 2: Data collection and visualization are important.
  – Chapter 3: Here’s the math you actually need to know.
  – Chapter 4: Time to unlearn what you thought you knew about statistics.
  – Chapter 5: You don’t understand your model until you can simulate from it.

• Part 2:
  – Chapter 6: Let’s think deeply about regression.
  – Chapter 7: You can’t just do regression, you have to understand regression.
  – Chapter 8: Least squares and all that.
  – Chapter 9: Let’s be clear about our uncertainty and about our prior knowledge.
  – Chapter 10: You don’t just fit models, you build models.
  – Chapter 11: Can you convince me to trust your model?
  – Chapter 12: Only fools work on the raw scale.

• Part 3:
  – Chapter 13: Modeling probabilities.
  – Chapter 14: Logistic regression pro tips.
  – Chapter 15: Building models from the inside out.

• Part 4:
  – Chapter 16: To understand the past, you must first know the future.
  – Chapter 17: Enough about your data. Tell me about the population.

• Part 5:
  – Chapter 18: How can flipping a coin help you estimate causal effects?
  – Chapter 19: Using correlation and assumptions to infer causation.
  – Chapter 20: Causal inference is just a kind of prediction.
  – Chapter 21: More assumptions, more problems.

• Part 6:
  – Chapter 22: Who’s got next?

• Appendixes:
  – Appendix A: R quick start.
  – Appendix B: These are our favorite workflow tips; what are yours?

In this book we present many methods and illustrate their use in many applications; we also try to give a sense of where these methods can fail, and we try to convey the excitement the first time that we learned about these ideas and applied them to our own problems.

Additional material for teaching and learning

Data for the examples and homework assignments; other teaching resources

The website www.stat.columbia.edu/~gelman/regression contains pointers to data and code for the examples and homework problems in the book, along with some teaching materials.
Prerequisites

This book does not require advanced mathematics. To understand the linear model in regression, you will need the algebra of the intercept and slope of a straight line, but it will not be necessary to follow the matrix algebra in the derivation of least squares computations. You will use exponents and logarithms at different points, especially in Chapters 12–15 in the context of nonlinear transformations and generalized linear models.

Software

Previous knowledge of programming is not required. You will do a bit of programming in the general-purpose statistical environment R when fitting and using the models in this book, and some of these fits will be performed using the Bayesian inference program Stan, which, like R, is free and open source. Readers new to R or to programming should first work their way through Appendix A.

We fit regressions using the `stan_glm` function in the `rstanarm` package in R, performing Bayesian inference using simulation. This is a slight departure from usual treatments of regression (including our earlier book), which use least squares and maximum likelihood, for example using the `lm` and `glm` functions in R. We discuss differences between these different software options, and between these different modes of inference, in Sections 1.6, 8.4, and 9.5. From the user’s perspective, switching to `stan_glm` doesn’t matter much except in making it easier to obtain probabilistic predictions and to propagate inferential uncertainty, and in certain problems with collinearity or sparse data (in which case the Bayesian approach in `stan_glm` gives more stable estimates), and when we wish to include prior information in the analysis. For most of the computations done in this book, similar results could be obtained using classical regression software if so desired.

Suggested courses

The material in this book can be broken up in several ways for one-semester courses. Here are some examples:

- **Basic linear regression**: Chapters 1–5 for review, then Chapters 6–9 (linear regression with one predictor) and Chapters 10–12 (multiple regression, diagnostics, and model building).
- **Applied linear regression**: Chapters 1–5 for review, then Chapters 6–12 (linear regression), Chapters 16–17 (design and poststratification), and selected material from Chapters 18–21 (causal inference) and Chapter 22 (advanced regression).
- **Applied regression and causal inference**: Quick review of Chapters 1–5, then Chapters 6–12 (linear regression), Chapter 13 (logistic regression), Chapters 16–17 (design and poststratification), and selected material from Chapters 18–21 (causal inference).
- **Causal inference**: Chapters 1, 7, 10, 11, and 13 for review of linear and logistic regression, then Chapters 18–21 in detail.
- **Generalized linear models**: Some review of Chapters 1–12, then Chapters 13–15 (logistic regression and generalized linear models), followed by selected material from Chapters 16–21 (design, poststratification, and causal inference) and Chapter 22 (advanced regression).

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