An Introduction to Gödel's Theorems

In 1931, the young Kurt Gödel published his First Incompleteness Theorem, which tells us that, for any sufficiently rich theory of arithmetic, there are some arithmetical truths the theory cannot prove. This remarkable result is among the most intriguing (and most misunderstood) in logic. Gödel also outlined an equally significant Second Incompleteness Theorem. How are these Theorems established, and why do they matter? Peter Smith answers these questions by presenting an unusual variety of proofs for the First Theorem, showing how to prove the Second Theorem, and exploring a family of related results (including some not easily available elsewhere). The formal explanations are interwoven with discussions of the wider significance of the two Theorems. This book – extensively rewritten for its second edition – will be accessible to philosophy students with a limited formal background. It is equally suitable for mathematics students taking a first course in mathematical logic.

PETER SMITH was formerly Senior Lecturer in Philosophy at the University of Cambridge. His books include *Explaining Chaos* (1998) and *An Introduction to Formal Logic* (2003), and he is also a former editor of the journal *Analysis*.

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Peter Smith
Frontmatter
More information

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Second edition

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For Patsy, as ever

Contents

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Preface

In 1931, the young Kurt Gödel published his First and Second Incompleteness Theorems; very often, these are referred to simply as 'Gödel's Theorems' (even though he proved many other important results). These Incompleteness Theorems settled – or at least, seemed to settle – some of the crucial questions of the day concerning the foundations of mathematics. They remain of the greatest significance for the philosophy of mathematics, though just what that significance is continues to be debated. It has also frequently been claimed that Gödel's Theorems have a much wider impact on very general issues about language, truth and the mind.

This book gives proofs of the Theorems and related formal results, and touches – necessarily briefly – on some of their implications. Who is the book for? Roughly speaking, for those who want a lot more fine detail than you get in books for a general audience (the best of those is Franzén, 2005), but who find the rather forbidding presentations in classic texts in mathematical logic (like Mendelson, 1997) too short on explanatory scene-setting. I assume only a modest amount of background in logic. So I hope philosophy students will find the book useful, as will mathematicians who want a more accessible exposition.

But don't be misled by the relatively relaxed style; don't try to browse through too quickly. We do cover a lot of ground in quite a bit of detail, and new ideas often come thick and fast. Take things slowly!

I originally intended to write a shorter book, leaving many of the formal details to be filled in from elsewhere. But while that plan might have suited some readers, I soon realized that it would seriously irritate others to be sent hither and thither to consult a variety of textbooks with different terminologies and different notations. So, in the end, I have given more or less full proofs of most of the key results we cover ('⊠' serves as our end-of-proof marker, as we want the more usual '□' for another purpose).

However, my original plan shows through in two ways. First, some proofs are still only partially sketched in. Second, I try to signal very clearly when the detailed proofs I do give can be skipped without much loss of understanding. With judicious skimming, you should be able to follow the main formal themes of the book even if you have limited taste for complex mathematical arguments. For those who want to fill in more details and test their understanding there are exercises on the book's website at www.godelbook.net, where there are also other supplementary materials.

As we go through, there is also an amount of broadly philosophical commentary. I follow Gödel in believing that our formal investigations and our general

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More information

reflections on foundational matters should illuminate and guide each other. I hope that these brief philosophical discussions – relatively elementary though certainly not always uncontentious – will also be reasonably widely accessible. Note however that I am more interested in patterns of ideas and arguments than in being historically very precise when talking e.g. about logicism or about Hilbert's Programme.

Writing a book like this presents many problems of organization. For example, we will need to call upon a number of ideas from the general theory of computation – we will make use of both the notion of a 'primitive recursive function' and the more general notion of a 'μ-recursive function'. Do we explain these related ideas all at once, up front? Or do we give the explanations many chapters apart, when the respective notions first get put to use?

I've mostly adopted the second policy, introducing new ideas as and when needed. This has its costs, but I think that there is a major compensating benefit, namely that the way the book is organized makes it clearer just what depends on what. It also reflects something of the historical order in which ideas emerged.

How does this second edition differ from the first? This edition is over twenty pages longer, but that isn't because there is much new material. Rather, I have mostly used the extra pages to make the original book more reader-friendly; there has been a lot of rewriting and rearrangement, particularly in the opening chapters. Perhaps the single biggest change is in using a more traditional line of proof for the adequacy of Robinson Arithmetic (Q) for capturing all the primitive recursive functions. I will probably have disappointed some readers by still resisting the suggestion that I provide a full-blown, warts-and-all, proof of the Second Theorem, though I do say rather more than before. But after all, this *is* supposed to be a relatively introductory book.

Below, I acknowledge the help that I have so generously been given by so many. But here I must express thanks of a quite different order to Patsy Wilson-Smith, without whose continuing love and support neither edition of this book would ever have been written. This book is for her.

Thanks

'Acknowledgements' is far too cold a word. I have acquired many intellectual debts in the course of writing this book: with great kindness, a lot of people have given me comments, suggestions and corrections. As a result, the book – whatever its remaining shortcomings – is so much better than it might have been.

My colleague Michael Potter has been an inspiring presence ever since I returned to Cambridge. Many thanks are due to him and to all those who gave me advice on draft extracts while I was writing the first edition, including the late Torkel Franzén, Tim Button, Luca Incurvati, Jeffrey Ketland, Aatu Koskensilta, Christopher Leary, Mary Leng, Toby Ord, Alex Paseau, Jacob Plotkin, José F. Ruiz, Kevin Scharp, Hartley Slater, and Tim Storer. I should especially mention Richard Zach, whose comments were particularly extensive and particularly helpful.

I no longer have a list of all those who found errors in the first printing. But Arnon Avron, Peter Milne, Saeed Salehi, and Adil Sanaulla prompted the most significant changes in content that made their way into corrected reprints of the first edition. Orlando May then spotted some still remaining technical errors, as well as a distressing number of residual typos. Jacob Plotkin, Tony Roy and Alfredo Tomasetta found further substantive mistakes. Again, I am very grateful.

It is an oddity that the books which are read the most – texts aimed at students – are reviewed the least in the journals: but Arnon Avron and Craig Smoryński did write friendly reviews of the first edition, and I have now tried to meet at least some of their expressed criticisms.

When I started working on this second edition, I posted some early parts on my blog Logic Matters, and received very useful suggestions from a number of people. Encouraged by that, at a late stage in the writing I experimentally asked for volunteers to proof-read thirty-page chunks of the book: the bribe I offered was tiny, just a mention here! But over forty more people took up the invitation, so every page was looked at again three or four times. Many of these readers applied themselves to the task with quite extraordinary care and attention, telling me not just about the inevitable typos, but about ill-phrased sentences, obscurities, phrases that puzzled a non-native speaker of English, and more besides. A handful of readers – I report this with mixed feelings – also found small technical errors *still* lurking in the text. The experiment, then, was a resounding success. So warm thanks are due to, among others, Sama Agahi, Amir Anvari, Bert Baumgaertner, Alex Blum, Seamus Bradley, Matthew Brammall, Benjamin Briggs,

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Catrin Campbell-Moore, Jordan Collins, Irena Cronin, Matthew Dentith, Neil Dewar, Jan Eißfeldt, Özge Ekin, Nicolas Fillion, Stefan Daniel Firth, Joshua P. Fry, Marc Alcobé García, André Gargoura, Jaime Gaspar, Jake Goritski, Scott Hendricks, Harrison Hibbert, Daniel Hoek, Anil Joshi, Abolfazl Karimi, Amir Khamseh, Baldur Arnviður Kristinsson, Jim Laird, S. P. Lam, Jim Lippard, Carl Mummett, Fredrick Nelson, Robert Rynasiewicz, Noah David Schweber, Alin Soare, Shane Steinert-Threlkeld, Andrew Stephenson, Alexander auf der Straße, S. P. Suresh, Andrew Tedder, Arhat Virdi, Benjamin Wells, John Wigglesworth, and Andrew Withy. I am very grateful to them all!

Three more readers require special mention. David Auerbach's many suggestions have uniformly made for significant improvements. Matthew Donald has also given me extremely helpful comments on the whole book. Frank Scheld has proof-read successive versions with enthusiasm and an eagle's eye. Again, very warm thanks.

Finally, I must thank Hilary Gaskin at Cambridge University Press, who initially accepted the book for publication and then offered me the chance to write a second edition.