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Introduction

This book is written for researchers and graduate students in the field of geometric mechanics, especially the theory of systems with symmetries. A wider audience might include differential geometers, algebraic geometers and singularity theorists. The aim of the book is to show that differential geometry in the sense of Sikorski is a powerful tool for the study of the geometry of spaces with singularities. We show that this understanding of differential geometry gives a complete description of the stratification structure of the space of orbits of a proper action of a connected Lie group G on a manifold P. We also show that the same approach can handle intersection singularities; see Section 8.2.

We assume here that the reader has a working knowledge of differential geometry and the topology of manifolds, and we use theorems in these fields freely without giving proofs or references. On the other hand, the material on differential spaces is developed from scratch. The results on differential spaces are proved in detail. This should make the book accessible to graduate students.

The book is split into two parts. In Part I, we introduce the reader to the differential geometry of singular spaces and prove some results, which are used in Part II to investigate concrete systems. The technique of differential geometry presented here is fairly straightforward, and the reader might get a false impression that the scope of the theory does not differ much from that of the geometry of manifolds. However, the examples given in Part I will serve as warnings that such an impression is false. Part II is devoted to applications of the general theory. Each chapter in this part may be considered as an extensive example of the use of differential geometry to deal with singularities in concrete problems. Since these problems occur in various theories, each chapter begins with a section introducing elements of the underlying theory, in order to show the reader the relevance of the problem under consideration.

The book contains no exercises, because the actual techniques involved are very simple. In addition to the standard techniques of the differential geometry of manifolds, we use techniques of algebraic geometry for rings of smooth functions. The fact that algebraically defined derivations of smooth functions admit integral curves is the main difference between differential and algebraic geometry.

The technical details of the presentation are based on the TEX style file chosen for the preparation of this book. Displayed results are labelled by the number of the chapter, the number of the section in the chapter and the number of the result within the section. For example, 'Lemma 2.1.3' stands for Lemma 1.3 in Chapter 2; it can also be read as the third lemma in Section 2.1. Displayed equations are referenced by the number of the chapter and the number of the equation within the chapter. For example, 'equation (3.21)' stands for equation 21 in Chapter 3.

This book is based on several years of research. Some of the results presented here were obtained by the author. Some other results have been taken directly from the work of other researchers. The remainder corresponds to an adaptation and reformulation of the work of other authors so that it fits into the theory presented here. In order to keep the flow of the presentation in the subsequent chapters free from obstructions, we give below a detailed description of the content of the book and the references to the literature.

Part I is devoted to a comprehensive presentation of the current status of the differential geometry of singular spaces. A comprehensive bibliography of the literature on differential spaces during the period 1965–1992 was published in 1993 by Buchner, Heller, Multarzyński and Sasin (Buchner *et al.*, 1993). According to these authors, the first paper on differential spaces was Sikorski (1967). In the same year, at a meeting of the American Mathematical Society, Aronszajn presented an extensive programme of differential-geometric study of subcartesian spaces in terms of singular charts. Aronszajn's subcartesian spaces included arbitrary subspaces of $\mathbb{R}^n$ (see Aronszajn, 1967). In 1973, Walczak showed that subcartesian spaces are special cases of differential spaces (see Walczak, 1973).

In Section 2.1, we describe the basic definitions and constructions of Sikorski's theory following his book (see Sikorski, 1972). The fundamental notion of this theory is the differential structure $C^\infty(S)$ of a space S, consisting of functions on S deemed to be smooth. The differential structure of a space carries all information about the geometry of the space. In particular, a map $\varphi : S \to T$ is smooth if it pulls back smooth functions to smooth functions. A diffeomorphism is an invertible smooth map with a smooth inverse. As in topology, subsets, products and quotients of differential spaces are differential spaces. However, the quotient differential space need not have the quotient topology. Proposition 2.1.11, which gives conditions for equivalence of the quotient differential-space topology and the quotient topology, is taken from the work of Pasternak-Winiarski (1984) .

A differential space S is subcartesian if every point of S has a neighbourhood diffeomorphic to a subset of some Cartesian space $\mathbb{R}^n$. The category of subcartesian differential spaces is the main object of our study. Manifolds are subcartesian spaces that are locally diffeomorphic to open subsets of $\mathbb{R}^n$. If M is a manifold, the collection of all local diffeomorphisms to open subsets of $\mathbb{R}^n$ forms the maximal atlas on M. Differential geometry, understood as the study of the geometry of a space in terms of the ring of smooth functions on that space, naturally extends from manifolds to subcartesian spaces. We do not go beyond subcartesian spaces, because a differential space which is not subcartesian need not have a locally finite dimension.

In Section 2.2, we show that subcartesian spaces admit partitions of unity. The importance of partitions of unity stems from the fact that they enable us to globalize collections of local data. The existence of partitions of unity on locally compact and paracompact differential spaces was first proved by Cegiełka (1974). Here, we follow the proof of Marshall (1975a).

In Chapter 3, we discuss vector fields on subcartesian spaces. A vector field on a manifold M can be described either as a derivation of a ring $C^\infty(M)$ of smooth functions on M or as a generator of a local one-parameter group of local diffeomorphisms of M. These two notions are equivalent if M is a manifold. However, they may be inequivalent on a subcartesian space S that is not a manifold.

In Section 3.1, we study the basic properties of derivations of the differential structure $C^\infty(S)$ of a subcartesian space S. We show that every derivation X of $C^\infty(S)$ can be locally extended to a derivation of $C^\infty(R^n)$. This result allows the study of ordinary differential equations on subcartesian spaces, which we discuss in Section 3.2. The existence and uniqueness theorem for integral curves of derivations on a subcartesian space was first proved by Śniatycki (2003a).

In Section 3.3, we discuss the tangent bundle space TS of S, defined as the space of derivations of $C^\infty(S)$ at points of S. In the literature, TS is also called the tangent pseudobundle or the Zariski tangent bundle. Following Watts (2006), we define the regular component S_{reg} of S as the set of all points p of S at which $\dim T_pS$ is locally constant, and prove that S_{reg} is open and dense in S and that the restriction TS_{reg} of TS to S_{reg} is locally spanned by global derivations; see Lusala *et al.* (2010). Example 3.3.12, taken from Epstein and

Śniatycki (2006), shows that a differential space that is regular everywhere need not be a manifold.

In Section 3.4, we study global derivations of S that generate local one-parameter groups of local diffeomorphisms. We call such global derivations vector fields. We show that the orbits of any family of vector fields on a subcartesian space S are smooth manifolds immersed in S. This result, first proved by Śniatycki (2003b), is a generalization of some theorems of Sussmann (1973) and Stefan (1974). In particular, it implies that orbits of the family $\mathfrak{X}(S)$ of all vector fields on S give a partition of S by smooth manifolds. Therefore, every subcartesian space S has a minimal partition by smooth manifolds. This result gives us an alternative interpretation of the strata of a minimal stratification of a subcartesian space, which we study in Chapter 4.

In Chapter 4, we discuss stratified spaces, first investigated by Whitney (1955), who called them 'manifold collections'. The term 'stratification' is due to Thom (1955–56). A stratified space is usually described as a topological space partitioned in a special way by smooth manifolds. Here, we restrict our considerations to stratified spaces that are also subcartesian differential spaces.

In Section 4.1, we discuss stratified subcartesian spaces following the work of Śniatycki (2003b) and Lusala and Śniatycki (2011). A stratified space is, by definition, partitioned by smooth manifolds. The results of Chapter 3 show that a subcartesian space is also partitioned by smooth manifolds, which are orbits of the family of all vector fields. We show that if a stratified space S is subcartesian and the stratification of S is locally trivial, then the partition of S by orbits of the family of all vector fields is also a stratification of S. Moreover, this second stratification of S is coarser than the original stratification. If the original stratification is minimal, then it is the same as the stratification given by the orbits of the family of all vector fields. In other words, a minimal locally trivial stratification of a subcartesian space is completely determined by its differential structure.

In Section 4.2, we describe the orbit type stratification $\mathfrak{M}$ of a manifold P given by a proper action on P of a connected Lie group G. This stratification is not minimal, because the union of all the strata is the manifold P. The presentation adopted here borrows from the presentations of the same topic in the books by Cushman and Bates (1997), Duistermaat and Kolk (2000), and Pflaum (2001).

Section 4.3 is devoted to a discussion of the structure of the orbit space $R = P/G$. We show that the projection to the orbit space R of the strata of $\mathfrak{M}$ is a locally trivial and minimal stratification of R. This is called the orbit type stratification of the orbit space R. We also show that R is a subcartesian space.

The material presented in Section 4.3 is based on the results of many authors. In particular, results of Bierstone (1975; 1980), Bochner's Linearization Theorem (Duistermaat and Kolk, 2000), the Hilbert–Weyl Theorem (Weyl, 1946), Palais's Slice Theorem (Palais, 1961), a theorem by Schwarz (1975) and the Tarski–Seidenberg Theorem (Abraham and Robbin, 1967). The form of presentation adopted here follows that of Cushman, Duistermaat and Śniatycki (2010). By combining the results of Sections 4.1 and 4.3, we conclude that the strata of the orbit type stratification of the orbit space R are orbits of the family of all vector fields on R. This result is the basis for the singular reduction of symmetries discussed in subsequent chapters.

In Section 4.4, we study a proper action of a Lie group on a locally compact subcartesian space. Palais's Slice Theorem applies to this case, and we prove that the space of orbits of the action is a locally compact differential space. We have no extension of Bochner's Linearization Theorem to subcartesian spaces, and we can prove neither that the orbit space is subcartesian nor that it is stratified. Nevertheless, the result obtained here suffices to prove singular reduction by stages in Section 6.5.

Chapter 5 is devoted to a discussion of differential forms on subcartesian spaces. We are led to three inequivalent notions of differential forms. Zariski differential forms on S are defined as alternating multilinear maps from spaces of pointwise derivations of $C^\infty(S)$ to real numbers. Zariski differential forms can be pulled back by smooth maps. If S is not a manifold, then exterior differentials of Zariski differential forms are not defined. The second possibility is Koszul differential forms, defined as alternating multilinear maps from spaces of global derivations of $C^\infty(S)$ to $C^\infty(S)$. We can take exterior differentials of Koszul forms, but we cannot define their pull-backs by differential maps. The third possibility is Marshall forms, which agree with Zariski forms and Koszul forms on the regular component S_{reg} of S. Marshall forms allow pull-backs, as well as exterior differentials. The presentation adopted here follows a paper by Marshall (1975a), Watts' theses (Watts, 2006; 2012) and his unpublished notes.

In Part II, we apply the general theory introduced in Part I to the problem of reduction of the symmetries of various systems. In most cases, we make an assumption that the action of the symmetry group G on the phase space P of the system is proper. This assumption implies that the orbit space P/G is stratified, and the study of reduction involves an investigation of the interplay between the stratification structure of P/G and the geometric structure characterizing the system under consideration.

There is no satisfactory theory of the structure of the space of orbits of an improper action of a Lie group on a manifold. However, if P is a symplectic manifold and the improper action of G on P is Hamiltonian, we can show that algebraic reduction, in terms of differential schemes, encodes a lot of information about the action of G on P. We also show that the information obtained by algebraic reduction may survive the process of quantization and may be decoded on the quantum level.

The objective of symplectic reduction, discussed in Chapter 6, is to describe the structure of the space of orbits of a Hamiltonian action of a connected Lie group G on a symplectic manifold (P, ω). For a proper action, we know that the orbit space $R = P/G$ is stratified, and we investigate the interaction between the stratification structure of R and the Poisson structure of R induced by the symplectic structure of P. We also discuss the case when the action of G on P fails to be proper.

In Section 6.1, we give a brief review of Hamiltonian actions of a Lie group G on a symplectic manifold (P, ω), the properties of the momentum map $J : P \to \mathfrak{g}^*$, and the Poisson algebra structure of $C^\infty(P)$ induced by the symplectic form ω on P. We begin with a discussion of the co-adjoint action of G on co-adjoint orbits in $\mathfrak{g}^*$ and describe the Kirillov–Kostant–Souriau symplectic form of a co-adjoint orbit (Kirillov, 1962; Kostant, 1966; Souriau, 1966). Moreover, we show that the momentum map for a co-adjoint orbit is the inclusion of the orbit in $\mathfrak{g}^*$. This introductory material is included here in order to establish the notation and to introduce the problem to readers who might be unfamiliar with symplectic geometry.

Symplectic reduction for a free and proper action was introduced by Meyer (1973) and Marsden and Weinstein (1974). It is known as regular reduction or Marsden–Weinstein reduction. The first study of the structure of the orbit space for a proper non-free Hamiltonian action of the symmetry group was the paper of Arms, Marsden and Moncrief (Arms *et al.*, 1981), who showed that the zero level of the momentum map is stratified.

The technique of singular reduction in terms of the Poisson algebra structure was initiated by Cushman (1983), and later formalized by Arms, Cushman and Gotay (Arms *et al.*, 1991). The role of Sikorski's theory of differential spaces in singular reduction was first described by Cushman and Śniatycki (2001). Comprehensive presentations of singular reduction have been given in the books by Cushman and Bates (1997) and Ortega and Ratiu (2004). Our discussion of singular reduction is contained in Sections 6.2–6.6. Our presentation differs from the presentations in Cushman and Bates (1997) and Ortega and Ratiu (2004) because we have the general theory developed in Part I at our disposal. Nevertheless, it has many points in common with earlier approaches.

In Section 6.2, we describe the structure of the orbit space $R = P/G$ in terms of the structure of the ring $C^\infty(R)$ of smooth functions on R. Using the results of Chapter 4, we describe strata of the orbit type stratifications of P/G as orbits of the family of all vector fields $\mathfrak{X}(R)$ on R. For each stratum of R, the Poisson structure on $C^\infty(R)$ induces the structure of a Poisson manifold. Since Poisson derivations of $C^\infty(R)$ are vector fields on R, orbits of the family $\mathfrak{P}(R)$ of all Poisson derivations of $C^\infty(R)$ give foliations of strata of R by symplectic leaves. A proof that a Poisson manifold is singularly foliated by symplectic leaves was given in the book by Libermann and Marle (1987).

In Section 6.3, we show that for each $\mu \in \mathfrak{g}^*$, the projection to R of the level set $J^{-1}(\mu)$ is a stratified space with symplectic strata, which are symplectomorphic to the corresponding symplectic leaves of strata of R. In Section 6.4, we obtain similar results for projections to R of $J^{-1}(O)$, provided that the co-adjoint orbit O is locally closed.[1] The main results obtained in Sections 6.3 and 6.4 are not new. However, the proofs of these results are new.

In Section 6.5, we apply the results of Section 4.4 to the case when the symmetry group G of (P, ω) has a normal subgroup H. In this case, we can first reduce the action of H. The result is a stratified Poisson space P/H with symmetry group G/H. Following Lusala and Śniatycki (to appear), we prove that the structure of the orbit space $(P/H)/(G/H)$ is isomorphic to that of P/G. This result is called 'reduction by stages' in the literature; see the book by Marsden, Misiołek, Ortega, Perlmutter and Ratiu (Marsden *et al.*, 2007).

In Section 6.6, we discuss the process of shifting, which gives rise to an equivalence between the reduction of $J^{-1}(O)$ and the reduction at zero for a shifted momentum map on $P \times O$, where O is a co-adjoint orbit. This is essential for the extension to non-zero co-adjoint orbits of the results on the commutation of quantization and reduction of $J^{-1}(0)$ discussed in the next chapter. Shifting was introduced for a free and proper action by Guillemin and Sternberg (1984). For a proper non-free action, shifting was first proved by Bates, Cushman, Hamilton and Śniatycki (Bates *et al.*, 2009).

In Section 6.7, we restrict singular reduction to the case when the action of G on P is free and proper. As a corollary, we obtain the results of the Marsden–Weinstein reduction (Marsden and Weinstein, 1974).

In Section 6.8, we discuss the case when the action of G on P is not proper. In this case, the ring of G-invariant functions on P need not separate the orbits, and singular reduction is not applicable. At present, there is no satisfactory theory of the structure of the space of orbits of an improper

[1] An example of a co-adjoint orbit which is not locally closed was first given by Pukanszky (1971). Here, we do not study such co-adjoint orbits; however, they were discussed by Ortega and Ratiu (2004).

action of a Lie group on a manifold. However, in our case, P is a symplectic manifold and the improper action of G on P is Hamiltonian, which allows algebraic reduction as discussed in Section 6.9. Algebraic reduction gives rise to a Poisson algebra defined in terms of differential schemes, which are differential-geometry analogues of schemes in algebraic geometry. The Poisson algebra of algebraic reduction encodes a lot of information about the action of G on P. The problem arises as to how to decode the information encoded in algebraic reduction and use it in applications. We return to this question in Chapter 7.

Algebraic reduction of the zero level of the momentum map was introduced by Śniatycki and Weinstein (1983). Algebraic reduction at non-zero co-adjoint orbits was introduced independently by Wilbour (1993) and Kimura (1993). Theorem 6.9.6 (the shifting theorem) was proved by Arms (1996). Example 6.9.4 was first investigated in the context of algebraic reduction by Arms, Gotay and Jennings (Arms *et al.*, 1990). Example 6.9.7 was first outlined in Śniatycki and Weinstein (1983); a full analysis of this example was given in Śniatycki (2005). Lemma 3.8.1 was proved by Bates (2007).

Chapter 7 is devoted to the problem of commutation of geometric quantization and reduction. The term 'geometric quantization' is used in mechanics and in representation theory. In both cases, it describes essentially the same mathematical procedure, but its starting points and aims are different in the two cases. In representation theory, quantization is a technique for obtaining a unitary representation of a connected Lie group from its action on a symplectic manifold. In quantum mechanics, geometric quantization provides a geometric way to transition from the classical to the quantum description of a physical system.

In physics, we often study a quantum subsystem of a classical system. This is usually done by starting with a classical description of the whole system and then imposing constraints to single out the subsystem, followed by reduction of spurious degrees of freedom and subsequent quantization. We expect that the physical results obtained will be the same as the results of a study of the subsystem in terms of quantization of the whole system. This expectation can be rephrased as the principle that quantization commutes with reduction.

The importance of commutation of quantization and reduction was realized in the study of the quantization of gauge theories and general relativity. According to Noether's Second Theorem (Noether, 1918), the presence of a gauge symmetry leads to a constraint in the theory, given by $J = 0$, where J is the momentum map for the gauge group action (Binz *et al.*, 2006). In the studies by Bleuler (1950) and Gupta (1950) of the quantization of electrodynamics, these authors quantized the full space of the Cauchy data for the electromagnetic field and imposed an appropriate constraint on the space of quantum states. On the other hand, Dirac's study of the quantization of gravity led to a distinction between first-class and second-class constraints (Dirac, 1964). First-class constraints could be imposed on the quantum level, whereas second-class constraints had to be imposed on the classical level.

It is rather difficult to give a definite answer in the framework of quantum field theory to the question of whether quantization and reduction commute. Guillemin and Sternberg (1982) proved that geometric quantization commutes with reduction provided that some strong technical assumptions are satisfied. Their approach was formulated in the framework of the representation theory of Lie groups. Geometric quantization has its roots in the work of Kirillov (1962), Auslander and Kostant (1971), Kostant (1966; 1970) and Souriau (1966). A comprehensive bibliography was given in a book by Woodhouse (1992).

We begin with a discussion of the significance of commutation of quantization and reduction in the framework of representation theory. In Section 7.1, we give a review of geometric quantization following Śniatycki (1980).

In Section 7.2, we discuss in general terms the problem of commutation of geometric quantization and singular reduction. This problem has been studied by Bates, Cushman, Hamilton and Śniatycki (Bates *et al.*, 2009), using an algebraic approach based on Śniatycki's earlier results on commutation of quantization and algebraic reduction (Śniatycki, 2012). The approach to the problem of commutation of geometric quantization and singular reduction, as well as many of the results presented in this section, is new.

In Section 7.3, we discuss various special cases. We begin with the case of a Kähler quantization of a compact symplectic manifold (P, ω) with a Hamiltonian action of a compact connected Lie group G, investigated by Guillemin and Sternberg (1982) and by Sjamaar (1995). We discuss which of the results of Guillemin and Sternberg and of Sjamaar follow from our general approach, and which of these results are specific to the approach that they used. Our results also hold when the symplectic manifold P and the Lie group G are not compact, and agree with the results of Huebschmann (2006). Next, we discuss conditions for commutation of singular reduction and quantization with respect to a real polarization. For a free and proper action of G on P, these conditions were first introduced by Śniatycki (1983), and subsequently studied by Duval, Elhadad, Gotay, Śniatycki and Tuynman; see Duval *et al.* (1990; 1991) and the references therein.

In Section 7.4, we discuss commutation of quantization and reduction at non-zero quantizable co-adjoint orbits using the shifting trick described in Section 6.6. The approach adopted here follows Śniatycki (2012).

In Section 7.5, we discuss the problem of commutation of geometric quantization and algebraic reduction. In fact, algebraic reduction was invented for this problem. In 1980, at a conference in Banff, Guillemin presented some unpublished results from his work with Sternberg. This lecture motivated the present author to investigate possible ways to generalize the results of Guillemin and Sternberg to singular momentum maps. In 1981, the author presented at a conference in Clausthal a paper discussing some examples in quantum mechanics which could be interpreted as quantum reduction of singular constraints (Śniatycki, 1983). Weinstein's reaction to this lecture led to a collaboration, which culminated in publication of a joint paper (Śniatycki and Weinstein, 1983). We discuss some special cases when the polarization is Kähler or real, and obtain results similar to the results for singular reduction. We conclude with some partial results on commutation of quantization and reduction for an improper action of the symmetry group.

Chapter 8 contains two more examples of reduction of symmetry. In Section 8.1, we discuss reduction of symmetry for a proper action of the symmetry group G of a non-holonomically constrained Hamiltonian system. We begin with a description of the distributional Hamiltonian formulation of constrained dynamics, following Bates and Śniatycki (1993). Next, we reformulate the distributional Hamiltonian formulation in terms of the almost-Poisson formulation of van der Schaft and Maschke (1994). This encodes the distributional Hamiltonian structure of the theory in the structure of $C^\infty(P)$. The space $C^\infty(P)^G$ of G-invariant functions is an almost-Poisson subalgebra of $C^\infty(P)$. Since the differential structure $C^\infty(P/G)$ of the orbit space P/G is isomorphic to $C^\infty(P)^G$, it inherits an almost-Poisson algebra structure, which was first used to discuss reduction by Koon and Marsden (1998).

The almost-Poisson bracket is a derivation and gives rise to a family $\mathfrak{P}(P/G)$ of almost-Poisson vector fields on P/G. The orbits of this family are manifolds. Each orbit Q carries a generalized distribution D_Q spanned by the restriction of $\mathfrak{P}(P/G)$ to Q. Moreover, D_Q carries a symplectic form ϖ_Q defined by the almost-Poisson structure of $C^\infty(Q)$. A comprehensive presentation of the current state of the geometry of non-holonomically constrained Hamiltonian systems can be found in a recent book by Cushman, Duistermaat and Śniatycki (Cushman *et al.*, 2010).

In Section 8.2, we discuss reduction of symmetries for a proper action of the symmetry group G of a Dirac structure. A Dirac structure on a manifold P is a maximal isotropic subbundle D of the Pontryagin bundle $P = TQ \times_Q T^*Q$