VARIATIONAL METHODS WITH APPLICATIONS IN SCIENCE AND ENGINEERING

There is an ongoing resurgence of applications in which the calculus of variations has direct relevance. *Variational Methods with Applications in Science and Engineering* reflects the strong connection between calculus of variations and the applications for which variational methods form the fundamental foundation. The material is presented in a manner that promotes development of an intuition about the concepts and methods with an emphasis on applications, and the priority of the application chapters is to provide a brief introduction to a variety of physical phenomena and optimization principles from a unified variational point of view. The first part of the book provides a modern treatment of the calculus of variations suitable for advanced undergraduate students and graduate students in applied mathematics, physical sciences, and engineering. The second part gives an account of several physical applications from a variational point of view, such as classical mechanics, optics and electromagnetics, modern physics, and fluid mechanics. A unique feature of this part of the text is derivation of the ubiquitous Hamilton's principle directly from the first law of thermodynamics, which enforces conservation of total energy, and the subsequent derivation of the governing equations of many discrete and continuous phenomena from Hamilton's principle. In this way, the reader will see how the traditional variational treatments of statics and dynamics are unified with the physics of fluids, electromagnetic fields, relativistic mechanics, and quantum mechanics through Hamilton's principle. The third part covers applications of variational methods to optimization and control of discrete and continuous systems, including image and data processing as well as numerical grid generation. The application chapters in parts two and three are largely independent of each other so that the instructor or reader can choose a path through the topics that aligns with their interests.

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To my father, Professor D. Wayne Cassel, for imparting a love of mathematics.

To Professor J. David A. Walker, for imparting a love of the application of mathematics to fluid mechanics.

To my wife Adrienne, and children Ryan, Nathan, and Sarah, for their unconditional love and support.

Contents

Preface

In a review of the book *Mathematics for Physics: A Guided Tour for Graduate Students* by Michael Stone and Paul Goldbart (2009), David Khmelnitskii perceptively writes:

> Without textbooks, the education of scientists is unthinkable. Textbook authors rearrange, repackage, and present established facts and discoveries – along the way straightening logic, excluding unnecessary details, and, finally, shrinking the volume of preparatory reading for the next generation. Writing them is therefore one of the most important collective tasks of the academic community, and an often underrated one at that. Textbooks are not easy to create, but once they are, the good ones become cornerstones, often advancing and redefining common knowledge.[1]

There is perhaps no other branch of applied mathematics that is more in need of such a "repackaging" than the calculus of variations.

A glance through the reference list at the end of the book will reveal that there are a number of now-classic texts on the calculus of variations from up through the mid-1960s, such as Weinstock (1952) and Gelfand and Fomin (1963), with a sharp drop subsequent to that period. The classic texts that emphasize applications, such as Morse and Feshback (1953) and Courant and Hilbert (1953), typically focus the majority of their discussion of variational methods on classical mechanics, for example, statics, dynamics, elasticity, and vibrations. Since that time, it has been more common to simply include the necessary elements of variational calculus in books dedicated to specific topics, such as analytical dynamics, dynamical systems, mechanical vibrations, elasticity, finite-element methods, and optimal control theory. More recently, the trend has been to avoid treating these subjects from a variational point of view altogether. Until very recently, therefore, modern stand-alone texts on the calculus of variations have been nearly nonexistent, and a small subset of modern texts in the above-mentioned subject areas include a limited treatment of variational calculus.

Not having had a course on the subject myself, I learned calculus of variations along with my students in a first-year graduate engineering analysis course that I first taught at the Illinois Institute of Technology (IIT) in 2001. The course covers matrices, eigenfunction theory, complex variable theory, and calculus

[1] *Physics Today*, October 2009.

of variations. It required several semesters of teaching the course for me to transition from the – unstated – attitude of "we all need to suffer through the calculus of variations for the sake of those students in solid mechanics, dynamics, and controls" to the – stated – attitude that "calculus of variations is one of the most widely applicable branches of mathematics for scientists and engineers!"

There has been a remarkable resurgence of applications in the last half century in which the calculus of variations has direct relevance. In addition to its traditional application to mechanics of solids and dynamics, it is now being applied in a variety of numerical methods, numerical grid generation, modern physics, various optimization settings, and fluid dynamics. Many of these applications, such as nonlinear optimal control theory applied to continuous systems, have only recently become tractable computationally with the advent of advanced algorithms and large computer systems. In my area of fluid mechanics, which has not traditionally been considered an area ripe for utilization of calculus of variations, there is increasing interest in applying optimal control theory to fluid mechanics and applying variational calculus to numerical methods and grid generation in computational fluid dynamics. With the growing interest in flow control, whereby small actuators are used to excite mechanisms within a flow to produce large changes in the overall flow structure, optimal control theory is making significant headway in providing a formal framework in which to consider various flow control techniques. In addition, numerical grid generation increasingly is being based on formal optimization principles and variational calculus rather than ad hoc techniques, such as elliptic grid generation. Variational calculus also impacts fluid mechanics through its applications to shape optimization and recent advances in hydrodynamic stability theory via transient-growth analysis.

Unfortunately, the majority of modern texts on applied mathematics do not reflect this revival of interest in variational methods. This is demonstrated by the fact that the genre of modern *advanced engineering mathematics* textbooks in wide use today typically do not include calculus of variations. See, for example, Kreyszig (2011), which is now in its 10th edition; O'Neil (2012); Zill and Wright (2014); Jeffrey (2002); and Greenberg (1998). Regrettably, calculus of variations has become too closely linked with certain application areas, such as mechanics of solids, analytical dynamics, and control theory. Such connections are reflected in many of the classic books on these subjects. This gives the impression that it is not a subject that has wide applicability in other areas of science and engineering. The paucity of current stand-alone textbooks on the calculus of variations reinforces the impression that the subject has matured and that there is little modern interest in the field.

Owing to the renaissance of applications in which variational methods have direct relevance and development of the computational resources required to treat them, there is need for a modern treatment of the subject from a general point of view that highlights the breadth of applications demonstrating the widespread applicability of variational methods. The present text is my humble attempt at providing such a treatment of variational methods. The objectives are twofold and include assisting the reader in *learning* and *applying* variational methods:

1. *Learn variational methods:* Part I provides a concise treatment of the fundamentals of the calculus of variations that are accessible to the typical student having a background in differential calculus and ordinary differential equations, but none in variational calculus.
2. *Apply variational methods:* Parts II and III provide a bridge between the fundamental material in Part I that is common to many areas of application and the area-specific applications. They provide an introduction to applications of calculus of variations in various fields and a link to dedicated texts and research literature in these subjects.

The content of the text reflects the strong connection between calculus of variations and the applications for which variational methods form the fundamental foundation. Most readers will be pleased to note that the mathematical fundamentals of calculus of variations (at least those necessary to pursue applications) are rather compact and are contained in a single chapter of the book. Therefore, the majority of the text consists of applications of variational calculus in various fields. The priority of these application chapters is to provide a brief introduction to a variety of physical phenomena and optimization principles from a unified variational point of view. The emphasis is on illustrating the wide range of applications of the calculus of variations, and the reader is referred to dedicated texts for more complete treatments of each topic. The centerpiece of these disparate subjects is *Hamilton's principle*, which provides a compact form of the dynamical equations of motion (its traditional area of application) and the governing equations for many other physical phenomena as illustrated throughout the text.

Given the emphasis on breadth of applications addressed using variational methods, it is necessary to sacrifice depth of treatment of each topic. This is the case not merely for space considerations but, more importantly, for clarity and unification of presentation. The objective in the application chapters is to provide a clear and concise introduction to the variational underpinnings of each field as the basis for further study and investigation. For example, optimization and control are being applied so broadly, including financial optimization, shape optimization, control of dynamical systems, grid generation, image processing, and so on, that it is instructive to view these topics within a common optimal control theory framework. It is also felt that there is value in readers with a background in a given field to be exposed to other related and complementary topics in order to see potential analogies in approach or opportunities for application of one topic in another.

A unique feature of the present text is the derivation of the ubiquitous Hamilton's principle directly from the first law of thermodynamics, which enforces conservation of total energy, and the subsequent derivation of the governing equations of many discrete and continuous phenomena from Hamilton's principle. In this way, the reader will see how the traditional variational treatments of statics and dynamics of discrete and continuous systems are unified with the physics of fluids, electromagnetic fields, relativistic mechanics, quantum mechanics, and so on through Hamilton's principle. I hope to impart to the reader some of the joy of discovering the interrelationship between these seemingly disparate physical principles that submit to the variational framework in general and the first law of thermodynamics through Hamilton's principle specifically.

In writing and arranging the text, I have made the following pedagogical assumptions:

1. Most readers are handicapped in learning a difficult topic unless they are persuaded of its relevance to their academic and/or professional development. That is, engineers and scientists must be convinced of the need for a subject or topic before learning it. It is insufficient to say (or imply), "Trust me; you need to know this." Although this should be a primary concern of instructors on the front lines of pedagogy, there is much that can be done in selecting and arranging the content of a textbook to assist both the instructor and the reader. For example, to motivate the engineering, scientific, and/or mathematical need for a subject or topic, one could include both broad and specific applications early in the development of each topic that students are, or will become, familiar with and/or provide historical motivations for development of a topic.
2. Engineers and scientists learn details better when they have an overall intuition or understanding on which to hang the details. This is why we typically feel like we have learned more about a topic when encountering it a second (or third, ...) time as compared to the first. Therefore, one must seek to provide and develop a physical and/or mathematical intuition that will inform theoretical developments, derivations, or proofs.
3. It is better to err on the side of too much detail in derivations and worked examples than too little. It is easier for the reader to skip a familiar step than to fill in an unfamiliar one. This is also done in order to clearly illustrate the concepts and methods developed in the text and to promote good problem-solving techniques with a minimum of "shortcuts."

This treatment is aimed at the advanced undergraduate or graduate engineering, physical sciences, or applied mathematics student. In addition, it could serve as a reference for researchers and practitioners in fields that are based on, or make use of, variational methods. The text assumes that readers are proficient with differential calculus, including vector calculus, and ordinary differential equations. Familiarity with basic matrix operations and partial differential equations is helpful, but not essential. Moreover, it is not necessary to have any background in the application areas treated in Chapters 4–12; however, such a background would certainly provide a useful perspective.

The book provides sufficient material to form the basis for a one-semester course on calculus of variations or a portion of an applied mathematics, mathematical physics, or engineering analysis course that includes such topics. The latter is the case at IIT, where this material is part of the engineering analysis course taken by many first-year graduate students in mechanical, aerospace, materials, chemical, civil, structural, electrical, and biomedical engineering. In addition, it could serve as a supplement or reference for courses in analytical dynamics, elasticity, mechanical vibrations, modern physics, fluid mechanics, optimal control theory, image processing, and so on that are taught from a variational point of view. Exercises are provided at the end of selected "fundamentals," or nonapplication, chapters, including each chapter in Part I and the first chapter in each of Parts II and III. Solutions are available to instructors at the book's website.

Because of the intended audience of the book, there is little emphasis on mathematical proofs. Instead, the material is presented in a manner that promotes development of an intuition about the concepts and methods with an emphasis on applications. Although primarily couched in terms of optimization theory, Luenberger (1969) does an excellent job of putting calculus of variations in the context of the underlying branches of mathematics known as linear vector spaces and functional analysis and is replete with useful geometric interpretations. Therefore, it provides an excellent complement to the present text for those with a more mathematical orientation than the present text typifies.

I would like to acknowledge certain individuals who have contributed to this book project. Professors Sudhakar Nair and Xiaoping Qian of IIT provided helpful comments and insights throughout development of the book. In particular, Professor Qian directed me to several useful references, and Professor Nair laid the foundation for the course from which this book had its start. Moreover, Professor Nair is largely responsible for keeping the mechanics tradition alive at IIT. I would also like to thank the reviewers who provided a host of useful comments that helped shape the emphasis and content of the book. The numerous students over the years whose questions have challenged me to continually improve the core content of the book are owed a special thank you; you are the ones that I had pictured in my mind as I penned each word and equation. In particular, I would like to acknowledge Mr. Jiacheng Wu for his insightful comments, probing questions, and thorough reading of the text. Finally, Mr. Peter Gordon of Cambridge University Press is to be thanked for his encouragement, advice, and editorial assistance throughout this book project. His "old-school" approach is refreshing in an age of increasingly detached publishers.

I would welcome any comments on the text from instructors and readers. I can be reached at cassel@iit.edu.