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Sets of Finite Perimeter and Geometric Variational Problems

An Introduction to Geometric Measure Theory

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Preface

Everyone talks about rock these days;
the problem is they forget about the roll.
Keith Richards

The theory of sets of finite perimeter provides, in the broader framework of Geometric Measure Theory (hereafter referred to as GMT), a particularly well-suited framework for studying the existence, symmetry, regularity, and structure of singularities of minimizers in those geometric variational problems in which surface area is minimized under a volume constraint. Isoperimetric-type problems constitute one of the oldest and more attractive areas of the Calculus of Variations, with a long and beautiful history, and a large number of still open problems and current research. The first aim of this book is to provide a pedagogical introduction to this subject, ranging from the foundations of the theory, to some of the most deep and beautiful results in the field, thus providing a complete background for research activity. We shall cover topics like the Euclidean isoperimetric problem, the description of geometric properties of equilibrium shapes for liquid drops and crystals, the regularity up to a singular set of codimension at least 8 for area minimizing boundaries, and, probably for the first time in book form, the theory of minimizing clusters developed (in a more sophisticated framework) by Almgren in his AMS Memoir [Alm76].

Ideas and techniques from GMT are of crucial importance also in the study of other variational problems (both of parametric and non-parametric character), as well as of partial differential equations. The secondary aim of this book is to provide a multi-leveled introduction to these tools and methods, by adopting an expository style which consists of both heuristic explanations and fully detailed technical arguments. In my opinion, among the various parts of GMT, the theory of sets of finite perimeter is the best suited for this aim. Compared to the theories of currents and varifolds, it uses a lighter notation and, virtually, no preliminary notions from Algebraic or Differential Geometry. At the same time, concerning, for example, key topics like partial regularity properties of minimizers and the analysis of their singularities, the deeper structure of many fundamental arguments can be fully appreciated in this simplified framework. Of course this line of thought has not to be pushed too far. But it is my conviction that a careful reader of this book will be able to enter other parts of GMT with relative ease, or to apply the characteristic tools of GMT in the study of problems arising in other areas of Mathematics.

The book is divided into four parts, which in turn are opened by rather detailed synopses. Depending on their personal backgrounds, different readers may like to use the book in different ways. As we shall explain in a moment, a short "crash-course" is available for complete beginners.

Part I contains the basic theory of Radon measures, Hausdorff measures, and rectifiable sets, and provides the background material for the rest of the book. I am not a big fan of "preliminary chapters", as they often miss a storyline, and quickly become boring. I have thus tried to develop Part I as independent, self-contained, and easily accessible reading. In any case, following the above mentioned "crash-course" makes it possible to see some action taking place without having to work through the entire set of preliminaries.

Part II opens with the basic theory of sets of finite perimeter, which is presented, essentially, as it appears in the original papers by De Giorgi [DG54, DG55, DG58]. In particular, we avoid the use of functions of bounded variation, hoping to better stimulate the development of a geometric intuition of the theory. We also present the original proof of De Giorgi's structure theorem, relying on Whitney's extension theorem, and avoiding the notion of rectifiable set. Later on, in the central portion of Part II, we make the theory of rectifiable sets from Part I enter into the game. We thus provide another justification of De Giorgi's structure theorem, and develop some crucial cut-and-paste competitors' building techniques, first and second variation formulae, and slicing formulae for boundaries. The methods and ideas introduced in this part are finally applied to study variational problems concerning confined liquid drops and anisotropic surface energies.

Part III deals with the regularity theory for local perimeter minimizers, as well as with the analysis of their singularities. In fact, we shall deal with the more general notion of (Λ, r_0)-perimeter minimizer, thus providing regularity results for several Plateau-type problems and isoperimetric-type problems. Finally, Part IV provides an introduction to the theory of minimizing clusters. These last two parts are definitely more advanced, and contain the deeper ideas and finer arguments presented in this book. Although their natural audience will unavoidably be made of more expert readers, I have tried to keep in these parts the same pedagogical point of view adopted elsewhere.

As I said, a "crash-course" on the theory of sets of finite perimeter, of about 130 pages, is available for beginners. The course starts with a revision of the basic theory of Radon measures, temporarily excluding differentiation theory (Chapters 1–4), plus some simple facts concerning weak gradients from Section 7.2. The notion of distributional perimeter is then introduced and used to prove the existence of minimizers in several variational problems, culminating with the solution of the Euclidean isoperimetric problem (Chapters 12–14). Finally, the differentiation theory for Radon measures is developed (Chapter 5), and then applied to clarify the geometric structure of sets of finite perimeter through the study of reduced boundaries (Chapter 15).

Each part is closed by a set of notes and remarks, mainly, but not only, of bibliographical character. The bibliographical remarks, in particular, are not meant to provide a complete picture of the huge literature on the problems considered in this book, and are limited to some suggestions for further reading. In a similar way, we now mention some monographs related to our subject.

Concerning Radon measures and rectifiable sets, further readings of exceptional value are Falconer [Fal86], Mattila [Mat95], and De Lellis [DL08].

For the classical approach to sets of finite perimeter in the context of functions of bounded variation, we refer readers to Giusti [Giu84], Evans and Gariepy [EG92], and Ambrosio, Fusco, and Pallara [AFP00].

The partial regularity theory of Part III does not follow De Giorgi's original approach [DG60], but it is rather modeled after the work of authors like Almgren, Allard, Bombieri, Federer, Schoen, Simon, etc. in the study of area minimizing currents and stationary varifolds. The resulting proofs only rely on direct comparison arguments and on geometrically viewable constructions, and should provide several useful reference points for studying more advanced regularity theories. Accounts and extensions of De Giorgi's original approach can be found in the monographs by Giusti [Giu84] and Massari and Miranda [MM84], as well as in Tamanini's beautiful lecture notes [Tam84].

Readers willing to enter into other parts of GMT have several choices. The introductory books by Almgren [Alm66] and Morgan [Mor09] provide initial insight and motivation. Suggested readings are then Simon [Sim83], Krantz and Parks [KP08], and Giaquinta, Modica, and Souček [GMS98a, GMS98b], as well as, of course, the historical paper by Federer and Fleming [FF60]. Concerning the regularity theory for minimizing currents, the paper by Duzaar and Steffen [DS02] is a valuable source for both its clarity and its completeness. Finally (and although, since its appearance, various crucial parts of the theory have found alternative, simpler justifications, and several major achievements have been obtained), Federer's legendary book [Fed69] remains the ultimate reference for many topics in GMT.

I wish to acknowledge the support received from several friends and colleagues in the realization of this project. This book originates from the lecture notes of a course that I held at the University of Duisburg-Essen in the Spring of 2005, under the advice of Sergio Conti. The successful use of these unpublished notes in undergraduate seminar courses by Peter Hornung and Stefan Müller convinced me to start the revision and expansion of their content. The work with Nicola Fusco and Aldo Pratelli on the stability of the Euclidean isoperimetric inequality [FMP08] greatly influenced the point of view on sets of finite perimeter adopted in this book, which has also been crucially shaped (particularly in connection with the regularity theory of Part III) by several, endless, mathematical discussions with Alessio Figalli. Alessio has also lectured at the University of Texas at Austin on a draft of the first three parts, supporting me with hundreds of comments. Another important contribution came from Guido De Philippis, who read the entire book *twice*, giving me much careful criticism and many useful suggestions. I was lucky to have the opportunity of discussing with Gian Paolo Leonardi various aspects of the theory of minimizing clusters presented in Part IV. Comments and errata were provided to me by Luigi Ambrosio (his lecture notes [Amb97] have been a major source of inspiration), Marco Cicalese, Matteo Focardi, Nicola Fusco, Frank Morgan, Matteo Novaga, Giovanni Pisante and Berardo Ruffini. Finally, I wish to thank Giovanni Alberti, Almut Burchard, Eric Carlen, Camillo de Lellis, Michele Miranda, Massimiliano Morini, and Emanuele Nunzio Spadaro for having expressed to me their encouragement and interest in this project.

I have the feeling that while I was busy trying to talk about the rock without forgetting about the roll, some errors and misprints made their way to the printed page. I will keep an errata list on my webpage.

This work was supported by the European Research Council through the Advanced Grant n. 226234 and the Starting Grant n. 258685, and was completed during my visit to the Department of Mathematics and the Institute for Computational Engineering and Sciences of the University of Texas at Austin. My thanks to the people working therein for the kind hospitality they have shown to me and my family.

Francesco Maggi

Notation

Notation 1 We work in the n**-dimensional Euclidean space** $\mathbb{R}^n$, that is the n-fold cartesian product of the space of real numbers $\mathbb{R}$. Therefore $x = (x_1, ..., x_n)$ is the generic element of $\mathbb{R}^n$, and $\{e_i\}_{i=1}^n$ is the **canonical orthonormal basis** of $\mathbb{R}^n$. We associate with $x \in \mathbb{R}^n \setminus \{0\}$ the one-dimensional linear subspace $\langle x \rangle$ of $\mathbb{R}^n$, $\langle x \rangle = \{tx : t \in \mathbb{R}\}$, called the **space spanned** by x. We endow $\mathbb{R}^n$ with the **Euclidean scalar product** $x \cdot y = \sum_{i=1}^n x_i y_i$. Given a linear subspace H of $\mathbb{R}^n$, we denote by $\dim(H)$ its dimension. If $\dim(H) = k$, then the **orthogonal space** to H in $\mathbb{R}^n$ is the $(n-k)$-dimensional linear space defined by

$$H^\perp = \Big\{ y \in \mathbb{R}^n : \text{if } x \in H \text{ then } y \cdot x = 0 \Big\},$$

and we set $x^\perp = \langle x \rangle^\perp$ for $x \neq 0$. The **Minkowski sum** of $E, F \subset \mathbb{R}^n$ is defined as

$$E + F = \Big\{ x + y : x \in E, \, y \in F \Big\},$$

with $x + F = \{x\} + F$ if $x \in \mathbb{R}^n$. A k**-dimensional plane** π in $\mathbb{R}^n$ is a set of the form $\pi = x + H$ where $x \in \mathbb{R}^n$ and H is a k-dimensional space in $\mathbb{R}^n$. When $k = 1$ we simply say that π is a **line** in $\mathbb{R}^n$. Given $E \subset \mathbb{R}^n$ and $\lambda > 0$ we set

$$\lambda E = \Big\{ \lambda x : x \in E \Big\}.$$

Defining the **Euclidean norm** $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$, the **Euclidean open ball** in $\mathbb{R}^n$ of center x and radius $r > 0$ is

$$B(x, r) = \Big\{ y \in \mathbb{R}^n : |y - x| < r \Big\}.$$

When $x = 0$ we set $B(0, r) = B_r$ and $B_1 = B$, so that $B(x, r) = x + B_r = x + r\,B$. We also set $S^{n-1} = \partial B = \{x \in \mathbb{R}^n : |x| = 1\}$ for the unit sphere in $\mathbb{R}^n$. Given $E, F \subset \mathbb{R}^n$, the **diameter of** E and the **distance between** E **and** F are

$$\operatorname{diam}(E) = \sup \Big\{ |x - y| : x, y \in E \Big\},$$
$$\operatorname{dist}(E, F) = \inf \Big\{ |x - y| : x \in E, \, y \in F \Big\}.$$

The **interior**, **closure**, and **topological boundary** (in the Euclidean topology) of $E \subset \mathbb{R}^n$ are denoted as usual as $\mathring{E}$, $\overline{E}$, and ∂E respectively. We write $E \subset\subset A$ and say E is **compactly contained** in A if $\overline{E} \subset A$.

Notation 2 A family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ is **disjoint** if $F_1, F_2 \in \mathcal{F}$, $F_1 \neq F_2$ implies $F_1 \cap F_2 = \emptyset$; it is **countable** if there exists a surjective function $f : \mathbb{N} \to \mathcal{F}$; it is a **covering** of $E \subset \mathbb{R}^n$ if $E \subset \bigcup_{F \in \mathcal{F}} F$. A **partition** of E is a disjoint covering of E which is composed of subsets of E.

Notation 3 (Linear functions) We denote by $\mathbb{R}^m \otimes \mathbb{R}^n$ the vector space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $T \in \mathbb{R}^m \otimes \mathbb{R}^n$, then $T(\mathbb{R}^n)$, the **image** of T, is a linear subspace of $\mathbb{R}^m$, and $\mathrm{Ker}\, T = \{T = 0\}$, the **kernel** of T, is a linear subspace of $\mathbb{R}^n$. The dimension of $T(\mathbb{R}^n)$ is called the **rank** of T, and T has **full rank** if $\dim(T(\mathbb{R}^n)) = m$. On $\mathbb{R}^m \otimes \mathbb{R}^n$ we define the **operator norm**,

$$\|T\| = \sup\left\{|Tx| : x \in \mathbb{R}^n, |x| < 1\right\}, \qquad T \in \mathbb{R}^m \otimes \mathbb{R}^n.$$

We notice that $\|T\| = \mathrm{Lip}(T)$, the Lipschitz constant of T on $\mathbb{R}^n$; see Chapter 7. If $T \in \mathbb{R}^m \otimes \mathbb{R}^n$, then we define a linear map $T^* \in \mathbb{R}^n \otimes \mathbb{R}^m$, called the **adjoint** of T, through the identity

$$(Tx) \cdot y = x \cdot (T^* y), \qquad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

Given $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, we define a linear map $w \otimes v$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, setting

$$(w \otimes v)x = (v \cdot x)w, \qquad x \in \mathbb{R}^n.$$

When $v \neq 0$ and $w \neq 0$ we say that $w \otimes v$ is a **rank-one map**, as we clearly have

$$(w \otimes v)(\mathbb{R}^n) = \langle w \rangle, \qquad \mathrm{Ker}\,(w \otimes v) = v^\perp.$$

We also notice the useful relations

$$(w \otimes v)^* = v \otimes w, \qquad \|w \otimes v\| = |v|\,|w|.$$

Rank-one maps induce a canonical identification of $\mathbb{R}^m \otimes \mathbb{R}^n$ with the space $\mathbb{R}^{m \times n}$ of $m \times n$ matrices $(a_{i,j})$ $(1 \le i \le n,\ 1 \le j \le m)$, having m rows and n columns. Indeed, if $V = \{v_j\}_{j=1}^n$ and $W = \{w_i\}_{i=1}^m$ are orthonormal bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, then, by definition of $w_i \otimes v_j$, we find that

$$T = \sum_{j=1}^n \sum_{i=1}^m \Big(w_i \cdot (Tv_j)\Big)\, w_i \otimes v_j.$$

Correspondingly, we associate T with the $m \times n$ matrix $(T_{i,j})$ with (i, j)th entry given by $T_{i,j} = w_i \cdot (Tv_j)$. When $n = m$, this identification allows us to define the notions of **determinant** and **trace** of a matrix for a linear map, by setting

$$\det T = \det(T_{i,j}), \qquad \mathrm{trace}\, T = \mathrm{trace}(T_{i,j}).$$

The functions $\det\colon \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$ and $\mathrm{trace}\colon \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$ are then independent of the choice of V underlying the identification of $\mathbb{R}^m \otimes \mathbb{R}^n$ with the space $\mathbb{R}^{m\times n}$, and inherit their usual properties. For example, we have

$$\det(TS) = \det(T)\det(S)\,, \qquad \forall T, S \in \mathbb{R}^n \otimes \mathbb{R}^n\,,$$

and $\det(\mathrm{Id}_n) = 1$, where of course $\mathrm{Id}_n x = x$ ($x \in \mathbb{R}^n$). If we denote by $\mathbf{GL}(n)$ the set of **invertible linear functions** $T \in \mathbb{R}^n \otimes \mathbb{R}^n$, then

$$\mathbf{GL}(n) = \left\{T \in \mathbb{R}^n \otimes \mathbb{R}^n : \det T \neq 0\right\}.$$

In particular, if $n \geq 2$ then $\det(w \otimes v) = 0$ for every $v, w \in \mathbb{R}^n$. The trace defines a linear function on $\mathbb{R}^n \otimes \mathbb{R}^n$ with $\mathrm{trace}(\mathrm{Id}_n) = n$ and, for every $T, S \in \mathbb{R}^n \otimes \mathbb{R}^n$,

$$\mathrm{trace}(T^*) = \mathrm{trace}(T)\,, \qquad \mathrm{trace}(TS) = \mathrm{trace}(ST)\,.$$

It is also useful to recall that for every $v, w \in \mathbb{R}^n$ we have

$$\mathrm{trace}(w \otimes v) = v \cdot w\,.$$

The trace operator can also be used to define a scalar product on $\mathbb{R}^m \otimes \mathbb{R}^n$:

$$T : S = \mathrm{trace}(S^*T) = \mathrm{trace}(T^*S)\,, \qquad T, S \in \mathbb{R}^m \otimes \mathbb{R}^n\,.$$

The norm corresponding to this scalar product (which does not coincide with the operator norm) is defined as

$$|T| = \sqrt{\mathrm{trace}(T^*T)}\,, \qquad T \in \mathbb{R}^m \otimes \mathbb{R}^n\,.$$

Notation 4 (Standard product decomposition of $\mathbb{R}^n$ into $\mathbb{R}^k \times \mathbb{R}^{n-k}$) When we need to decompose $\mathbb{R}^n$ as the cartesian product $\mathbb{R}^k \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$, we denote by $\mathbf{p}\colon \mathbb{R}^n \to \mathbb{R}^k \times \{0\} = \mathbb{R}^k$ and $\mathbf{q}\colon \mathbb{R}^n \to \{0\} \times \mathbb{R}^{n-k} = \mathbb{R}^{n-k}$ the horizontal and vertical projections, so that $x = (\mathbf{p}x, \mathbf{q}x)$, $x \in \mathbb{R}^n$. We then introduce the cylinder of center $x \in \mathbb{R}^n$ and radius $r > 0$,

$$\mathbf{C}(x, r) = \left\{y \in \mathbb{R}^n : |\mathbf{p}(y - x)| < r\,, |\mathbf{q}(y - x)| < r\right\},$$

and the k-dimensional ball of center $z \in \mathbb{R}^k$ and radius $r > 0$,

$$\mathbf{D}(z, r) = \left\{w \in \mathbb{R}^k : |z - w| < r\right\}.$$

Moreover, we always abbreviate

$$\mathbf{C}(0, r) = \mathbf{C}_r\,, \qquad \mathbf{C}_1 = \mathbf{C}\,, \qquad \mathbf{D}(0, r) = \mathbf{D}_r\,, \qquad \mathbf{D}_1 = \mathbf{D}\,.$$

When $k = n - 1$, we alternatively set $\mathbf{p}x = x'$ and $\mathbf{q}x = x_n$, so that $x = (x', x_n)$. Correspondingly we denote the gradient operator in $\mathbb{R}^n$ and in $\mathbb{R}^{n-1}$, respectively, by ∇ and $\nabla' = (\partial_1, ..., \partial_{n-1})$. If $u\colon \mathbb{R}^n \to \mathbb{R}$ has gradient $\nabla u(x) \in \mathbb{R}^n$ at $x \in \mathbb{R}^n$, then we set $\nabla u(x) = (\nabla' u(x), \partial_n u(x))$.