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978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory

Francesco Maggi

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SETS OF FINITE PERIMETER AND GEOMETRIC VARIATIONAL PROBLEMS

The marriage of analytic power to geometric intuition drives many of today's mathematical advances, yet books that build the connection from an elementary level remain scarce. This engaging introduction to geometric measure theory bridges analysis and geometry, taking readers from basic theory to some of the most celebrated results in modern analysis.

The theory of sets of finite perimeter provides a simple and effective framework. Topics covered include existence, regularity, analysis of singularities, characterization, and symmetry results for minimizers in geometric variational problems, starting from the basics about Hausdorff measures in Euclidean spaces, and ending with complete proofs of the regularity of area-minimizing hypersurfaces up to singular sets of codimension (at least) 8.

Explanatory pictures, detailed proofs, exercises, and remarks providing heuristic motivation and summarizing difficult arguments make this graduate-level textbook suitable for self-study and also a useful reference for researchers. Readers require only undergraduate analysis and basic measure theory.

Francesco Maggi is an Associate Professor at the Università degli Studi di Firenze, Italy.

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**Sets of Finite Perimeter and Geometric
Variational Problems**
An Introduction to Geometric Measure Theory

FRANCESCO MAGGI

Università degli Studi di Firenze, Italy



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Frontmatter

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Francesco Maggi

Frontmatter

[More information](#)

Contents

	<i>Preface</i>	page xiii
	<i>Notation</i>	xvii
	PART I RADON MEASURES ON \mathbb{R}^n	1
1	Outer measures	4
	1.1 Examples of outer measures	4
	1.2 Measurable sets and σ -additivity	7
	1.3 Measure Theory and integration	9
2	Borel and Radon measures	14
	2.1 Borel measures and Carathéodory's criterion	14
	2.2 Borel regular measures	16
	2.3 Approximation theorems for Borel measures	17
	2.4 Radon measures. Restriction, support, and push-forward	19
3	Hausdorff measures	24
	3.1 Hausdorff measures and the notion of dimension	24
	3.2 \mathcal{H}^1 and the classical notion of length	27
	3.3 $\mathcal{H}^n = \mathcal{L}^n$ and the isodiametric inequality	28
4	Radon measures and continuous functions	31
	4.1 Lusin's theorem and density of continuous functions	31
	4.2 Riesz's theorem and vector-valued Radon measures	33
	4.3 Weak-star convergence	41
	4.4 Weak-star compactness criteria	47
	4.5 Regularization of Radon measures	49
5	Differentiation of Radon measures	51
	5.1 Besicovitch's covering theorem	52

Cambridge University Press

978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory

Francesco Maggi

Frontmatter

[More information](#)

viii

Contents

5.2	Lebesgue–Besicovitch differentiation theorem	58
5.3	Lebesgue points	62
6	Two further applications of differentiation theory	64
6.1	Campanato’s criterion	64
6.2	Lower dimensional densities of a Radon measure	66
7	Lipschitz functions	68
7.1	Kirszbraun’s theorem	69
7.2	Weak gradients	72
7.3	Rademacher’s theorem	74
8	Area formula	76
8.1	Area formula for linear functions	77
8.2	The role of the singular set $Jf = 0$	80
8.3	Linearization of Lipschitz immersions	82
8.4	Proof of the area formula	84
8.5	Area formula with multiplicities	85
9	Gauss–Green theorem	89
9.1	Area of a graph of codimension one	89
9.2	Gauss–Green theorem on open sets with C^1 -boundary	90
9.3	Gauss–Green theorem on open sets with almost C^1 -boundary	93
10	Rectifiable sets and blow-ups of Radon measures	96
10.1	Decomposing rectifiable sets by regular Lipschitz images	97
10.2	Approximate tangent spaces to rectifiable sets	99
10.3	Blow-ups of Radon measures and rectifiability	102
11	Tangential differentiability and the area formula	106
11.1	Area formula on surfaces	106
11.2	Area formula on rectifiable sets	108
11.3	Gauss–Green theorem on surfaces	110
	Notes	114
	PART II SETS OF FINITE PERIMETER	117
12	Sets of finite perimeter and the Direct Method	122
12.1	Lower semicontinuity of perimeter	125
12.2	Topological boundary and Gauss–Green measure	127
12.3	Regularization and basic set operations	128
12.4	Compactness from perimeter bounds	132

Cambridge University Press

978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory

Francesco Maggi

Frontmatter

[More information](#)

<i>Contents</i>		ix
12.5	Existence of minimizers in geometric variational problems	136
12.6	Perimeter bounds on volume	141
13	The coarea formula and the approximation theorem	145
13.1	The coarea formula	145
13.2	Approximation by open sets with smooth boundary	150
13.3	The Morse–Sard lemma	154
14	The Euclidean isoperimetric problem	157
14.1	Steiner inequality	158
14.2	Proof of the Euclidean isoperimetric inequality	165
15	Reduced boundary and De Giorgi’s structure theorem	167
15.1	Tangential properties of the reduced boundary	171
15.2	Structure of Gauss–Green measures	178
16	Federer’s theorem and comparison sets	183
16.1	Gauss–Green measures and set operations	184
16.2	Density estimates for perimeter minimizers	189
17	First and second variation of perimeter	195
17.1	Sets of finite perimeter and diffeomorphisms	196
17.2	Taylor’s expansion of the determinant close to the identity	198
17.3	First variation of perimeter and mean curvature	200
17.4	Stationary sets and monotonicity of density ratios	204
17.5	Volume-constrained perimeter minimizers	208
17.6	Second variation of perimeter	211
18	Slicing boundaries of sets of finite perimeter	215
18.1	The coarea formula revised	215
18.2	The coarea formula on \mathcal{H}^{n-1} -rectifiable sets	223
18.3	Slicing perimeters by hyperplanes	225
19	Equilibrium shapes of liquids and sessile drops	229
19.1	Existence of minimizers and Young’s law	230
19.2	The Schwartz inequality	237
19.3	A constrained relative isoperimetric problem	242
19.4	Liquid drops in the absence of gravity	247
19.5	A symmetrization principle	250
19.6	Sessile liquid drops	253
20	Anisotropic surface energies	258
20.1	Basic properties of anisotropic surface energies	258
20.2	The Wulff problem	262

Cambridge University Press

978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory

Francesco Maggi

Frontmatter

[More information](#)

x

Contents

20.3	Reshetnyak's theorems	269
	Notes	272
PART III REGULARITY THEORY AND ANALYSIS OF SINGULARITIES		
21	(Λ, r_0)-perimeter minimizers	278
21.1	Examples of (Λ, r_0) -perimeter minimizers	278
21.2	(Λ, r_0) and local perimeter minimality	280
21.3	The $C^{1,\gamma}$ -regularity theorem	282
21.4	Density estimates for (Λ, r_0) -perimeter minimizers	282
21.5	Compactness for sequences of (Λ, r_0) -perimeter minimizers	284
22	Excess and the height bound	290
22.1	Basic properties of the excess	291
22.2	The height bound	294
23	The Lipschitz approximation theorem	303
23.1	The Lipschitz graph criterion	303
23.2	The area functional and the minimal surfaces equation	305
23.3	The Lipschitz approximation theorem	308
24	The reverse Poincaré inequality	320
24.1	Construction of comparison sets, part one	324
24.2	Construction of comparison sets, part two	329
24.3	Weak reverse Poincaré inequality	332
24.4	Proof of the reverse Poincaré inequality	334
25	Harmonic approximation and excess improvement	337
25.1	Two lemmas on harmonic functions	338
25.2	The “excess improvement by tilting” estimate	340
26	Iteration, partial regularity, and singular sets	345
26.1	The $C^{1,\gamma}$ -regularity theorem in the case $\Lambda = 0$	345
26.2	The $C^{1,\gamma}$ -regularity theorem in the case $\Lambda > 0$	351
26.3	$C^{1,\gamma}$ -regularity of the reduced boundary, and the characterization of the singular set	354
26.4	C^1 -convergence for sequences of (Λ, r_0) -perimeter minimizers	355
27	Higher regularity theorems	357
27.1	Elliptic equations for derivatives of Lipschitz minimizers	357
27.2	Some higher regularity theorems	359

Cambridge University Press

978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory

Francesco Maggi

Frontmatter

[More information](#)

	<i>Contents</i>	xi
28	Analysis of singularities	362
	28.1 Existence of densities at singular points	364
	28.2 Blow-ups at singularities and tangent minimal cones	366
	28.3 Simons' theorem	372
	28.4 Federer's dimension reduction argument	375
	28.5 Dimensional estimates for singular sets	379
	28.6 Examples of singular minimizing cones	382
	28.7 A Bernstein-type theorem	385
	Notes	386
	PART IV MINIMIZING CLUSTERS	391
29	Existence of minimizing clusters	398
	29.1 Definitions and basic remarks	398
	29.2 Strategy of proof	402
	29.3 Nucleation lemma	406
	29.4 Truncation lemma	408
	29.5 Infinitesimal volume exchanges	410
	29.6 Volume-fixing variations	414
	29.7 Proof of the existence of minimizing clusters	424
30	Regularity of minimizing clusters	431
	30.1 Infiltration lemma	431
	30.2 Density estimates	435
	30.3 Regularity of planar clusters	437
	Notes	444
	<i>References</i>	445
	<i>Index</i>	453

Cambridge University Press

978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An
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Francesco Maggi

Frontmatter

[More information](#)

Cambridge University Press

978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory

Francesco Maggi

Frontmatter

[More information](#)

Preface

*Everyone talks about rock these days;
the problem is they forget about the roll.*

Keith Richards

The theory of sets of finite perimeter provides, in the broader framework of Geometric Measure Theory (hereafter referred to as GMT), a particularly well-suited framework for studying the existence, symmetry, regularity, and structure of singularities of minimizers in those geometric variational problems in which surface area is minimized under a volume constraint. Isoperimetric-type problems constitute one of the oldest and more attractive areas of the Calculus of Variations, with a long and beautiful history, and a large number of still open problems and current research. The first aim of this book is to provide a pedagogical introduction to this subject, ranging from the foundations of the theory, to some of the most deep and beautiful results in the field, thus providing a complete background for research activity. We shall cover topics like the Euclidean isoperimetric problem, the description of geometric properties of equilibrium shapes for liquid drops and crystals, the regularity up to a singular set of codimension at least 8 for area minimizing boundaries, and, probably for the first time in book form, the theory of minimizing clusters developed (in a more sophisticated framework) by Almgren in his AMS Memoir [Alm76].

Ideas and techniques from GMT are of crucial importance also in the study of other variational problems (both of parametric and non-parametric character), as well as of partial differential equations. The secondary aim of this book is to provide a multi-leveled introduction to these tools and methods, by adopting an expository style which consists of both heuristic explanations and fully detailed technical arguments. In my opinion, among the various parts of GMT,

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978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory

Francesco Maggi

Frontmatter

[More information](#)

xiv

Preface

the theory of sets of finite perimeter is the best suited for this aim. Compared to the theories of currents and varifolds, it uses a lighter notation and, virtually, no preliminary notions from Algebraic or Differential Geometry. At the same time, concerning, for example, key topics like partial regularity properties of minimizers and the analysis of their singularities, the deeper structure of many fundamental arguments can be fully appreciated in this simplified framework. Of course this line of thought has not to be pushed too far. But it is my conviction that a careful reader of this book will be able to enter other parts of GMT with relative ease, or to apply the characteristic tools of GMT in the study of problems arising in other areas of Mathematics.

The book is divided into four parts, which in turn are opened by rather detailed synopses. Depending on their personal backgrounds, different readers may like to use the book in different ways. As we shall explain in a moment, a short “crash-course” is available for complete beginners.

Part I contains the basic theory of Radon measures, Hausdorff measures, and rectifiable sets, and provides the background material for the rest of the book. I am not a big fan of “preliminary chapters”, as they often miss a storyline, and quickly become boring. I have thus tried to develop Part I as independent, self-contained, and easily accessible reading. In any case, following the above mentioned “crash-course” makes it possible to see some action taking place without having to work through the entire set of preliminaries.

Part II opens with the basic theory of sets of finite perimeter, which is presented, essentially, as it appears in the original papers by De Giorgi [DG54, DG55, DG58]. In particular, we avoid the use of functions of bounded variation, hoping to better stimulate the development of a geometric intuition of the theory. We also present the original proof of De Giorgi’s structure theorem, relying on Whitney’s extension theorem, and avoiding the notion of rectifiable set. Later on, in the central portion of Part II, we make the theory of rectifiable sets from Part I enter into the game. We thus provide another justification of De Giorgi’s structure theorem, and develop some crucial cut-and-paste competitors’ building techniques, first and second variation formulae, and slicing formulae for boundaries. The methods and ideas introduced in this part are finally applied to study variational problems concerning confined liquid drops and anisotropic surface energies.

Part III deals with the regularity theory for local perimeter minimizers, as well as with the analysis of their singularities. In fact, we shall deal with the more general notion of (Λ, r_0) -perimeter minimizer, thus providing regularity results for several Plateau-type problems and isoperimetric-type problems. Finally, Part IV provides an introduction to the theory of minimizing clusters. These last two parts are definitely more advanced, and contain the deeper ideas

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Francesco Maggi

Frontmatter

[More information](#)*Preface*

xv

and finer arguments presented in this book. Although their natural audience will unavoidably be made of more expert readers, I have tried to keep in these parts the same pedagogical point of view adopted elsewhere.

As I said, a “crash-course” on the theory of sets of finite perimeter, of about 130 pages, is available for beginners. The course starts with a revision of the basic theory of Radon measures, temporarily excluding differentiation theory (Chapters 1–4), plus some simple facts concerning weak gradients from Section 7.2. The notion of distributional perimeter is then introduced and used to prove the existence of minimizers in several variational problems, culminating with the solution of the Euclidean isoperimetric problem (Chapters 12–14). Finally, the differentiation theory for Radon measures is developed (Chapter 5), and then applied to clarify the geometric structure of sets of finite perimeter through the study of reduced boundaries (Chapter 15).

Each part is closed by a set of notes and remarks, mainly, but not only, of bibliographical character. The bibliographical remarks, in particular, are not meant to provide a complete picture of the huge literature on the problems considered in this book, and are limited to some suggestions for further reading. In a similar way, we now mention some monographs related to our subject.

Concerning Radon measures and rectifiable sets, further readings of exceptional value are Falconer [Fal86], Mattila [Mat95], and De Lellis [DL08].

For the classical approach to sets of finite perimeter in the context of functions of bounded variation, we refer readers to Giusti [Giu84], Evans and Gariépy [EG92], and Ambrosio, Fusco, and Pallara [AFP00].

The partial regularity theory of Part III does not follow De Giorgi’s original approach [DG60], but it is rather modeled after the work of authors like Almgren, Allard, Bombieri, Federer, Schoen, Simon, etc. in the study of area minimizing currents and stationary varifolds. The resulting proofs only rely on direct comparison arguments and on geometrically viewable constructions, and should provide several useful reference points for studying more advanced regularity theories. Accounts and extensions of De Giorgi’s original approach can be found in the monographs by Giusti [Giu84] and Massari and Miranda [MM84], as well as in Tamanini’s beautiful lecture notes [Tam84].

Readers willing to enter into other parts of GMT have several choices. The introductory books by Almgren [Alm66] and Morgan [Mor09] provide initial insight and motivation. Suggested readings are then Simon [Sim83], Krantz and Parks [KP08], and Giaquinta, Modica, and Souček [GMS98a, GMS98b], as well as, of course, the historical paper by Federer and Fleming [FF60]. Concerning the regularity theory for minimizing currents, the paper by Duzaar and Steffen [DS02] is a valuable source for both its clarity and its completeness. Finally (and although, since its appearance, various crucial parts of the theory

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978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory

Francesco Maggi

Frontmatter

[More information](#)

xvi

Preface

have found alternative, simpler justifications, and several major achievements have been obtained), Federer's legendary book [Fed69] remains the ultimate reference for many topics in GMT.

I wish to acknowledge the support received from several friends and colleagues in the realization of this project. This book originates from the lecture notes of a course that I held at the University of Duisburg-Essen in the Spring of 2005, under the advice of Sergio Conti. The successful use of these unpublished notes in undergraduate seminar courses by Peter Hornung and Stefan Müller convinced me to start the revision and expansion of their content. The work with Nicola Fusco and Aldo Pratelli on the stability of the Euclidean isoperimetric inequality [FMP08] greatly influenced the point of view on sets of finite perimeter adopted in this book, which has also been crucially shaped (particularly in connection with the regularity theory of Part III) by several, endless, mathematical discussions with Alessio Figalli. Alessio has also lectured at the University of Texas at Austin on a draft of the first three parts, supporting me with hundreds of comments. Another important contribution came from Guido De Philippis, who read the entire book *twice*, giving me much careful criticism and many useful suggestions. I was lucky to have the opportunity of discussing with Gian Paolo Leonardi various aspects of the theory of minimizing clusters presented in Part IV. Comments and errata were provided to me by Luigi Ambrosio (his lecture notes [Amb97] have been a major source of inspiration), Marco Cicalese, Matteo Focardi, Nicola Fusco, Frank Morgan, Matteo Novaga, Giovanni Pisante and Berardo Ruffini. Finally, I wish to thank Giovanni Alberti, Almut Burchard, Eric Carlen, Camillo de Lellis, Michele Miranda, Massimiliano Morini, and Emanuele Nunzio Spadaro for having expressed to me their encouragement and interest in this project.

I have the feeling that while I was busy trying to talk about the rock without forgetting about the roll, some errors and misprints made their way to the printed page. I will keep an errata list on my webpage.

This work was supported by the European Research Council through the Advanced Grant n. 226234 and the Starting Grant n. 258685, and was completed during my visit to the Department of Mathematics and the Institute for Computational Engineering and Sciences of the University of Texas at Austin. My thanks to the people working therein for the kind hospitality they have shown to me and my family.

Francesco Maggi

Cambridge University Press

978-1-107-02103-7 - Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory

Francesco Maggi

Frontmatter

[More information](#)

Notation

Notation 1 We work in the n -dimensional Euclidean space \mathbb{R}^n , that is the n -fold cartesian product of the space of real numbers \mathbb{R} . Therefore $x = (x_1, \dots, x_n)$ is the generic element of \mathbb{R}^n , and $\{e_i\}_{i=1}^n$ is the **canonical orthonormal basis** of \mathbb{R}^n . We associate with $x \in \mathbb{R}^n \setminus \{0\}$ the one-dimensional linear subspace $\langle x \rangle$ of \mathbb{R}^n , $\langle x \rangle = \{tx : t \in \mathbb{R}\}$, called the **space spanned** by x . We endow \mathbb{R}^n with the **Euclidean scalar product** $x \cdot y = \sum_{i=1}^n x_i y_i$. Given a linear subspace H of \mathbb{R}^n , we denote by $\dim(H)$ its dimension. If $\dim(H) = k$, then the **orthogonal space** to H in \mathbb{R}^n is the $(n - k)$ -dimensional linear space defined by

$$H^\perp = \{y \in \mathbb{R}^n : \text{if } x \in H \text{ then } y \cdot x = 0\},$$

and we set $x^\perp = \langle x \rangle^\perp$ for $x \neq 0$. The **Minkowski sum** of $E, F \subset \mathbb{R}^n$ is defined as

$$E + F = \{x + y : x \in E, y \in F\},$$

with $x + F = \{x\} + F$ if $x \in \mathbb{R}^n$. A k -dimensional plane π in \mathbb{R}^n is a set of the form $\pi = x + H$ where $x \in \mathbb{R}^n$ and H is a k -dimensional space in \mathbb{R}^n . When $k = 1$ we simply say that π is a **line** in \mathbb{R}^n . Given $E \subset \mathbb{R}^n$ and $\lambda > 0$ we set

$$\lambda E = \{\lambda x : x \in E\}.$$

Defining the **Euclidean norm** $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$, the **Euclidean open ball** in \mathbb{R}^n of center x and radius $r > 0$ is

$$B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

When $x = 0$ we set $B(0, r) = B_r$ and $B_1 = B$, so that $B(x, r) = x + B_r = x + rB$. We also set $S^{n-1} = \partial B = \{x \in \mathbb{R}^n : |x| = 1\}$ for the unit sphere in \mathbb{R}^n . Given $E, F \subset \mathbb{R}^n$, the **diameter of** E and the **distance between** E and F are

$$\text{diam}(E) = \sup\{|x - y| : x, y \in E\},$$

$$\text{dist}(E, F) = \inf\{|x - y| : x \in E, y \in F\}.$$

The **interior**, **closure**, and **topological boundary** (in the Euclidean topology) of $E \subset \mathbb{R}^n$ are denoted as usual as $\overset{\circ}{E}$, \overline{E} , and ∂E respectively. We write $E \subset\subset A$ and say E is **compactly contained** in A if $\overline{E} \subset A$.

Notation 2 A family \mathcal{F} of subsets of \mathbb{R}^n is **disjoint** if $F_1, F_2 \in \mathcal{F}$, $F_1 \neq F_2$ implies $F_1 \cap F_2 = \emptyset$; it is **countable** if there exists a surjective function $f: \mathbb{N} \rightarrow \mathcal{F}$; it is a **covering** of $E \subset \mathbb{R}^n$ if $E \subset \bigcup_{F \in \mathcal{F}} F$. A **partition** of E is a disjoint covering of E which is composed of subsets of E .

Notation 3 (Linear functions) We denote by $\mathbb{R}^m \otimes \mathbb{R}^n$ the vector space of linear maps from \mathbb{R}^n to \mathbb{R}^m . If $T \in \mathbb{R}^m \otimes \mathbb{R}^n$, then $T(\mathbb{R}^n)$, the **image** of T , is a linear subspace of \mathbb{R}^m , and $\text{Ker } T = \{T = 0\}$, the **kernel** of T , is a linear subspace of \mathbb{R}^n . The dimension of $T(\mathbb{R}^n)$ is called the **rank** of T , and T has **full rank** if $\dim(T(\mathbb{R}^n)) = m$. On $\mathbb{R}^m \otimes \mathbb{R}^n$ we define the **operator norm**,

$$\|T\| = \sup \{ |Tx| : x \in \mathbb{R}^n, |x| < 1 \}, \quad T \in \mathbb{R}^m \otimes \mathbb{R}^n.$$

We notice that $\|T\| = \text{Lip}(T)$, the Lipschitz constant of T on \mathbb{R}^n ; see Chapter 7. If $T \in \mathbb{R}^m \otimes \mathbb{R}^n$, then we define a linear map $T^* \in \mathbb{R}^n \otimes \mathbb{R}^m$, called the **adjoint** of T , through the identity

$$(Tx) \cdot y = x \cdot (T^*y), \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

Given $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, we define a linear map $w \otimes v$ from \mathbb{R}^n to \mathbb{R}^m , setting

$$(w \otimes v)x = (v \cdot x)w, \quad x \in \mathbb{R}^n.$$

When $v \neq 0$ and $w \neq 0$ we say that $w \otimes v$ is a **rank-one map**, as we clearly have

$$(w \otimes v)(\mathbb{R}^n) = \langle w \rangle, \quad \text{Ker}(w \otimes v) = v^\perp.$$

We also notice the useful relations

$$(w \otimes v)^* = v \otimes w, \quad \|w \otimes v\| = |v| |w|.$$

Rank-one maps induce a canonical identification of $\mathbb{R}^m \otimes \mathbb{R}^n$ with the space $\mathbb{R}^{m \times n}$ of $m \times n$ matrices $(a_{i,j})$ ($1 \leq i \leq m$, $1 \leq j \leq n$), having m rows and n columns. Indeed, if $V = \{v_j\}_{j=1}^n$ and $W = \{w_i\}_{i=1}^m$ are orthonormal bases of \mathbb{R}^n and \mathbb{R}^m respectively, then, by definition of $w_i \otimes v_j$, we find that

$$T = \sum_{j=1}^n \sum_{i=1}^m (w_i \cdot (Tv_j)) w_i \otimes v_j.$$

Correspondingly, we associate T with the $m \times n$ matrix $(T_{i,j})$ with (i, j) th entry given by $T_{i,j} = w_i \cdot (Tv_j)$. When $n = m$, this identification allows us to define the notions of **determinant** and **trace** of a matrix for a linear map, by setting

$$\det T = \det(T_{i,j}), \quad \text{trace } T = \text{trace}(T_{i,j}).$$

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Frontmatter

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The functions $\det: \mathbb{R}^n \otimes \mathbb{R}^n \rightarrow \mathbb{R}$ and $\text{trace}: \mathbb{R}^n \otimes \mathbb{R}^n \rightarrow \mathbb{R}$ are then independent of the choice of V underlying the identification of $\mathbb{R}^m \otimes \mathbb{R}^n$ with the space $\mathbb{R}^{m \times n}$, and inherit their usual properties. For example, we have

$$\det(TS) = \det(T) \det(S), \quad \forall T, S \in \mathbb{R}^n \otimes \mathbb{R}^n,$$

and $\det(\text{Id}_n) = 1$, where of course $\text{Id}_n x = x$ ($x \in \mathbb{R}^n$). If we denote by $\mathbf{GL}(n)$ the set of **invertible linear functions** $T \in \mathbb{R}^n \otimes \mathbb{R}^n$, then

$$\mathbf{GL}(n) = \{T \in \mathbb{R}^n \otimes \mathbb{R}^n : \det T \neq 0\}.$$

In particular, if $n \geq 2$ then $\det(w \otimes v) = 0$ for every $v, w \in \mathbb{R}^n$. The trace defines a linear function on $\mathbb{R}^n \otimes \mathbb{R}^n$ with $\text{trace}(\text{Id}_n) = n$ and, for every $T, S \in \mathbb{R}^n \otimes \mathbb{R}^n$,

$$\text{trace}(T^*) = \text{trace}(T), \quad \text{trace}(TS) = \text{trace}(ST).$$

It is also useful to recall that for every $v, w \in \mathbb{R}^n$ we have

$$\text{trace}(w \otimes v) = v \cdot w.$$

The trace operator can also be used to define a scalar product on $\mathbb{R}^m \otimes \mathbb{R}^n$:

$$T : S = \text{trace}(S^*T) = \text{trace}(T^*S), \quad T, S \in \mathbb{R}^m \otimes \mathbb{R}^n.$$

The norm corresponding to this scalar product (which does not coincide with the operator norm) is defined as

$$|T| = \sqrt{\text{trace}(T^*T)}, \quad T \in \mathbb{R}^m \otimes \mathbb{R}^n.$$

Notation 4 (Standard product decomposition of \mathbb{R}^n into $\mathbb{R}^k \times \mathbb{R}^{n-k}$) When we need to decompose \mathbb{R}^n as the cartesian product $\mathbb{R}^k \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$, we denote by $\mathbf{p}: \mathbb{R}^n \rightarrow \mathbb{R}^k \times \{0\} = \mathbb{R}^k$ and $\mathbf{q}: \mathbb{R}^n \rightarrow \{0\} \times \mathbb{R}^{n-k} = \mathbb{R}^{n-k}$ the horizontal and vertical projections, so that $x = (\mathbf{p}x, \mathbf{q}x)$, $x \in \mathbb{R}^n$. We then introduce the cylinder of center $x \in \mathbb{R}^n$ and radius $r > 0$,

$$\mathbf{C}(x, r) = \{y \in \mathbb{R}^n : |\mathbf{p}(y-x)| < r, |\mathbf{q}(y-x)| < r\},$$

and the k -dimensional ball of center $z \in \mathbb{R}^k$ and radius $r > 0$,

$$\mathbf{D}(z, r) = \{w \in \mathbb{R}^k : |z-w| < r\}.$$

Moreover, we always abbreviate

$$\mathbf{C}(0, r) = \mathbf{C}_r, \quad \mathbf{C}_1 = \mathbf{C}, \quad \mathbf{D}(0, r) = \mathbf{D}_r, \quad \mathbf{D}_1 = \mathbf{D}.$$

When $k = n-1$, we alternatively set $\mathbf{p}x = x'$ and $\mathbf{q}x = x_n$, so that $x = (x', x_n)$. Correspondingly we denote the gradient operator in \mathbb{R}^n and in \mathbb{R}^{n-1} , respectively, by ∇ and $\nabla' = (\partial_1, \dots, \partial_{n-1})$. If $u: \mathbb{R}^n \rightarrow \mathbb{R}$ has gradient $\nabla u(x) \in \mathbb{R}^n$ at $x \in \mathbb{R}^n$, then we set $\nabla u(x) = (\nabla' u(x), \partial_n u(x))$.