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Francesco Maggi
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PART ONE

Radon measures on $\mathbb{R}^n$

Synopsis

In this part we discuss the basic theory of Radon measures on $\mathbb{R}^n$. Roughly speaking, if $\mathcal{P}(\mathbb{R}^n)$ denotes the set of the parts of $\mathbb{R}^n$, then a Radon measure μ on $\mathbb{R}^n$ is a function $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$, which is countably additive (at least) on the family of Borel sets of $\mathbb{R}^n$, takes finite values on bounded sets, and is completely identified by its values on open sets. The Lebesgue measure on $\mathbb{R}^n$ and the Dirac measure δ_x at $x \in \mathbb{R}^n$ are well-known examples of Radon measures on $\mathbb{R}^n$. Moreover, any locally summable function on $\mathbb{R}^n$, as well as any k-dimensional surface in $\mathbb{R}^n$, $1 \le k \le n-1$, can be naturally identified with a Radon measure on $\mathbb{R}^n$. There are good reasons to look at such familiar objects from this particular point of view. Indeed, the natural notion of convergence for sequences of Radon measures satisfies very flexible compactness properties. As a consequence, the theory of Radon measures provides a unified framework for dealing with the various convergence and compactness phenomena that one faces in the study of geometric variational problems. For example, a sequence of continuous functions on $\mathbb{R}^n$ that (as a sequence of Radon measures) is converging to a surface in $\mathbb{R}^n$ is something that cannot be handled with the notions of convergence usually considered on spaces of continuous functions or on Lebesgue spaces. Similarly, the existence of a tangent plane to a surface at one of its points can be understood as the convergence of the (Radon measures naturally associated with) re-scaled and translated copies of the surface to the (Radon measure naturally associated with the) tangent plane itself. This peculiar point of view opens the door for a geometrically meaningful (and analytically powerful) extension of the notion of differentiability to the wide class of objects, the family of rectifiable sets, that one must consider in solving geometric variational problems.

Part I is divided into two main portions. The first one (Chapters 1–6) is devoted to the more abstract aspects of the theory. In Chapters 1–4, we introduce the main definitions, present the most basic examples, and prove the fundamental representation and compactness theorems about Radon measures. (These results already suffice to give an understanding of the basic theory of sets of finite perimeter as presented in the first three chapters of Part II.) Differentiation

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theory, and its applications, are discussed in Chapters 5–6. In the second portion of Part I (Chapters 7–11), we consider Radon measures from a more geometric viewpoint, focusing on the interaction between Euclidean geometry and Measure Theory, and covering topics such as Lipschitz functions, Hausdorff measures, area formulae, rectifiable sets, and measure-theoretic differentiability. These are prerequisites to more advanced parts of the theory of sets of finite perimeter, and can be safely postponed until really needed. We now examine more closely each chapter.

In Chapters 1–2 we introduce the notions of Borel and Radon measure. This is done in the context of *outer measures*, rather than in the classical context of standard measures defined on σ-algebras. We simultaneously develop both the basic properties relating Borel and Radon measures to the Euclidean topology of $\mathbb{R}^n$ and the basic examples of the theory that are obtained by combining the definitions of Lebesgue and Hausdorff measures with the operations of restriction to a set and push-forward through a function.

In Chapter 3 we look more closely at Hausdorff measures. We establish their most basic properties and introduce the notion of Hausdorff dimension. Next, we show equivalence between the Lebesgue measure on $\mathbb{R}^n$ and the n-dimensional Hausdorff measure on $\mathbb{R}^n$, and we study the relation between the elementary notion of length of a curve, based on the existence of a parametrization, and the notion induced by one-dimensional Hausdorff measures.

In Chapter 4 we further develop the general theory of Radon measures. In particular, the deep link between Radon measures and continuous functions with compact support is presented, leading to the definition of *vector-valued* Radon measures, of weak-star convergence of Radon measures, and to the proof of the fundamental Riesz's representation theorem: every bounded linear functional on $C^0_c(\mathbb{R}^n;\mathbb{R}^m)$ is representable as integration with respect to an $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$. This last result, in turn, is the key to the weak-star compactness criterion for sequences of Radon measures.

Chapters 5–6 present differentiation theory and its applications. The goal is to compare two Radon measures ν and μ by looking, as $r \to 0^+$, at the ratios

$$\frac{\nu(B(x,r))}{\mu(B(x,r))},$$

which are defined at those x where μ is supported (i.e., $\mu(B(x,r)) > 0$ for every $r > 0$). The Besicovitch–Lebesgue differentiation theorem ensures that, for μ-a.e. x in the support of μ, these ratios converge to a finite limit $u(x)$, and that restriction of ν to the support of μ equals integration of u with respect to μ. Differentiation theory plays a crucial role in proving the validity of classical (or generalized) differentiability properties in many situations.

In Chapter 7 we study the basic properties of Lipschitz functions, proving Rademacher's theorem about the almost everywhere classical differentiability of Lipschitz functions, and Kirszbraun's theorem concerning the optimal extension problem for vector-valued Lipschitz maps.

Chapter 8 presents the area formula, which relates the Hausdorff measure of a set in $\mathbb{R}^n$ with that of its Lipschitz images into any $\mathbb{R}^m$ with $m \geq n$. As a consequence, the classical notion of area of a k-dimensional surface M in $\mathbb{R}^n$ is seen to coincide with the k-dimensional Hausdorff measure of M. Some applications of the area formula are presented in Chapter 9, where, in particular, the classical Gauss–Green theorem is proved.

In Chapter 10 we introduce one of the most important notions of Geometric Measure Theory, that of a k-dimensional rectifiable set in $\mathbb{R}^n$ $(1 \leq k \leq n-1)$. This is a very broad generalization of the concept of k-dimensional C^1-surface, allowing for complex singularities but, at the same time, retaining tangential differentiability properties, at least in a measure-theoretic sense. A crucial result is the following: if the k-dimensional blow-ups of a Radon measure μ converge to k-dimensional linear spaces (seen as Radon measures), then it turns out that μ itself is the restriction of the k-dimensional Hausdorff measure to a k-dimensional rectifiable set.

In Chapter 11, we introduce the notion of tangential differentiability of a Lipschitz function with respect to a rectifiable set, extend the area formula to this context, and prove the divergence theorem on C^2-surfaces with boundary.

1

Outer measures

Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all subsets of $\mathbb{R}^n$. An **outer measure** μ on $\mathbb{R}^n$ is a set function on $\mathbb{R}^n$ with values in $[0,\infty]$, $\mu\colon \mathcal{P}(\mathbb{R}^n) \to [0,\infty]$, with $\mu(\emptyset) = 0$ and

$$E \subset \bigcup_{h\in\mathbb{N}} E_h \quad \Rightarrow \quad \mu(E) \le \sum_{h\in\mathbb{N}} \mu(E_h)\,.$$

This property, called **σ-subadditivity**, implies the **monotonicity** of μ,

$$E \subset F \quad \Rightarrow \quad \mu(E) \le \mu(F)\,.$$

1.1 Examples of outer measures

Simple familiar examples of outer measures are the Dirac measure and the counting measure. The **Dirac measure** δ_x at $x \in \mathbb{R}^n$ is defined on $E \subset \mathbb{R}^n$ as

$$\delta_x(E) = \begin{cases} 1\,, & x \in E\,,\\ 0\,, & x \notin E\,,\end{cases}$$

while the **counting measure** # of E is

$$\#(E) = \begin{cases} \text{number of elements of } E\,, & \text{if } E \text{ is finite}\,,\\ +\infty\,, & \text{if } E \text{ is infinite}\,.\end{cases}$$

The two most important examples of outer measures are Lebesgue and Hausdorff measures.

Lebsegue measure: The **Lebesgue measure** of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{L}^n(E) = \inf_{\mathcal{F}} \sum_{Q\in\mathcal{F}} r(Q)^n\,,$$

where $\mathcal{F}$ is a countable covering of E by cubes with sides parallel to the coordinate axes, and $r(Q)$ denotes the side length of Q (the cubes Q are not assumed to be open, nor closed). The Lebesgue measure $\mathcal{L}^n(E)$ is interpreted as the n-dimensional volume of E. Usually, we write

$$\mathcal{L}^n(E) = |E|\,,$$

and refer to $|E|$ as the *volume* of E. Clearly, $\mathcal{L}^n$ is an outer measure. Moreover, it is translation-invariant, that is $|x + E| = |E|$ for every $x \in \mathbb{R}^n$, and satisfies the scaling law $|\lambda E| = \lambda^n |E|$, $\lambda > 0$. If $B = \{x \in \mathbb{R}^n : |x| < 1\}$ is the Euclidean unit ball of $\mathbb{R}^n$, then we set $\omega_n = |B|$. It is easily seen that $\omega_1 = 2$.

Hausdorff measure: Let $n, k \in \mathbb{N}$, with $n \geq 2$ and $1 \leq k \leq n - 1$. A bounded open set $A \subset \mathbb{R}^k$ and a function $f \in C^1(\mathbb{R}^k; \mathbb{R}^n)$ define a **k-dimensional parametrized surface** $f(A)$ in $\mathbb{R}^n$ provided f is injective on A with $Jf(x) > 0$ for every $x \in A$. Here $Jf(x)$ denotes the **Jacobian** of f at x, namely

$$Jf(x) = \sqrt{\det(\nabla f(x)^* \nabla f(x))}\,,$$

where, if $k = 1$, this means that $Jf(x) = |f'(x)|$. The condition $Jf(x) > 0$ ensures that $\nabla f(x)(\mathbb{R}^k)$ is a k-dimensional subspace of $\mathbb{R}^n$. The k-dimensional area of $f(A)$ is then classically defined as

$$k\text{-dimensional area of } f(A) = \int_A Jf(x)\,dx\,. \qquad (1.1)$$

In the study of geometric variational problems we need to extend this definition of k-dimensional area to more general sets than k-dimensional C^1-images. Hausdorff measures are introduced to this end. To avoid the use of parametrizations the definition is based on a covering procedure, as in the construction of the Lebesgue measure. Given $n, k \in \mathbb{N}$, $\delta > 0$, the **k-dimensional Hausdorff measure of step** δ of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{H}^k_\delta(E) = \inf_{\mathcal{F}} \sum_{F \in \mathcal{F}} \omega_k \Big(\frac{\operatorname{diam}(F)}{2}\Big)^k\,, \qquad (1.2)$$

where $\mathcal{F}$ is a countable covering of E by sets $F \subset \mathbb{R}^n$ such that $\operatorname{diam}(F) < \delta$; see Figure 1.1. The **$k$-dimensional Hausdorff measure** of E is then

$$\mathcal{H}^k(E) = \sup_{\delta \in (0,\infty]} \mathcal{H}^k_\delta(E) = \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(E)\,. \qquad (1.3)$$

It is trivial to see that, for every $\delta \in (0, \infty]$, $\mathcal{H}^k_\delta$ is an outer measure. As an immediate consequence, $\mathcal{H}^k$ is an outer measure too. In a similar way one proves that $\mathcal{H}^k$ is translation-invariant and that it satisfies the scaling law $\mathcal{H}^k(\lambda E) = \lambda^k \mathcal{H}^k(E)$, $\lambda > 0$. The fact that $\mathcal{H}^k(f(A))$ agrees with the classical notion of area on a k-dimensional parametrized surface $f(A)$ as defined in (1.1) is the content of the important *area formula*, discussed in Chapter 8.

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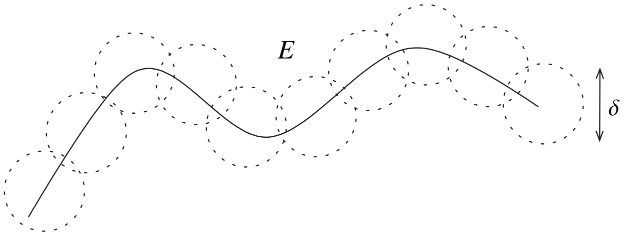

Figure 1.1 When computing $\mathcal{H}^k_\delta(E)$ one sums up, corresponding to each element F of a covering $\mathcal{F}$ of E, the k-dimensional measure of a k-dimensional ball of diameter $\mathrm{diam}(F)$. The minimization process used to compute $\mathcal{H}^k_\delta(E)$ does not detect any "deviation from straightness" of E taking place at a scale smaller than δ; see also Remark 1.1. Hence, one takes the limit $\delta \to 0^+$.

Remark 1.1 The idea behind the definition of Hausdorff measures is readily understood by considering the following statements concerning the case $k = 1$, $n = 2$ (see Chapter 3 for proofs).

(i) If E is a segment, then, for every $\delta > 0$, $\mathcal{H}^1_\delta(E)$ and $\mathcal{H}^1(E)$ coincide with the Euclidean length of E. If E is a polygonal curve composed of finitely many segments of length at least d, then, for every $\delta \in (0, d)$, $\mathcal{H}^1_\delta(E)$ and $\mathcal{H}^1(E)$ both agree with the Euclidean length of E.

(ii) If E is a curve of diameter d and $\delta \ge d$, then $\mathcal{H}^1_\delta(E) \le d$ (use the covering $\mathcal{F} = \{E\}$ of E in (1.2)), while, of course, the length of E can be arbitrarily large. It is only in the limit $\delta \to 0^+$ that $\mathcal{H}^1_\delta(E)$ approaches the length of E; see Section 3.2.

(iii) If E is countable (hence, zero-dimensional), then $\mathcal{H}^1(E) = 0$.

(iv) If E is an open set of $\mathbb{R}^2$ (i.e., a two-dimensional set), then $\mathcal{H}^1(E) = \infty$.

Remark 1.2 Given $s \in [0, \infty)$, the s-dimensional Hausdorff measures $\mathcal{H}^s_\delta$ and $\mathcal{H}^s$ are defined by simply replacing k with s in (1.2) and (1.3). The normalization constant ω_k is replaced by

$$\omega_s = \frac{\pi^{s/2}}{\Gamma(1 + s/2)}\,, \qquad s \ge 0\,,$$

where $\Gamma\colon (0, \infty) \to [1, \infty)$ is the Euler Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \mathrm{d}t\,, \qquad s > 0\,.$$

This is consistent as $\omega_k = \pi^{k/2}\Gamma(1+k/2)^{-1}$ for $k \in \mathbb{N}$, $k \ge 1$. Once again $\mathcal{H}^s_\delta$ and $\mathcal{H}^s$ are translation-invariant outer measures, with $\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$, $\lambda > 0$.

Exercise 1.3 Clearly, in the definition of $\mathcal{H}^s_\delta(E)$, we may equivalently consider coverings of E by subsets of E. Similarly,

(i) we may use coverings of E by closed convex sets intersecting E: indeed, the diameter of a set is the same as the diameter of its closed convex hull, and, if a set in $\mathcal{F}$ does not intersect E, it is convenient to discard it;
(ii) we may use coverings of E by open sets intersecting E: indeed, for every $F \subset \mathbb{R}^n$ and $\varepsilon > 0$, the **ε-neighborhood of** F,

$$I_\varepsilon(F) = \Big\{x \in \mathbb{R}^n : \operatorname{dist}(x, F) < \varepsilon\Big\}, \tag{1.4}$$

is open, contains F, and is such that $\operatorname{diam}(F_\varepsilon) \le \operatorname{diam}(F) + 2\varepsilon$.

1.2 Measurable sets and σ-additivity

Given a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$, we say that the outer measure μ on $\mathbb{R}^n$ is **σ-additive** on $\mathcal{F}$, provided

$$\mu\Big(\bigcup_{h\in\mathbb{N}} E_h\Big) = \sum_{h\in\mathbb{N}} \mu(E_h),$$

for every disjoint sequence $\{E_h\}_{h\in\mathbb{N}} \subset \mathcal{F}$ (i.e., $E_h \cap E_k = \emptyset$ if $h \ne k$). Accordingly to our naive intuition about the notion of measure, we would expect any reasonable measure to be σ-additive on $\mathcal{P}(\mathbb{R}^n)$. However, this fails even in the case of the Lebesgue measure $\mathcal{L}^1$ on $\mathbb{R}$. To show this, let us consider the classical **Vitali's example**. Define an equivalence relation $\approx$ on $(0, 1)$, so that $x \approx y$ if and only if $x - y$ is rational. By the axiom of choice, there exists a set $E \subset (0, 1)$ containing exactly one element from each of the equivalence classes defined by $\approx$ on $(0, 1)$. If $\{x_h\}_{h\in\mathbb{N}} = \mathbb{Q} \cap (0, 1)$, then the sequence of sets

$$E_h = \Big(x_h + \big(E \cap (0, 1 - x_h)\big)\Big) \cup \Big((x_h - 1) + \big(E \cap (1 - x_h, 1)\big)\Big)$$

is, by construction of E, disjoint. By the translation invariance of $\mathcal{L}^1$,

$$|E_h| = |E \cap (0, 1 - x_h)| + |E \cap (1 - x_h, 1)| = |E|,$$

with $(0, 1) = \bigcup_{h\in\mathbb{N}} E_h$. The σ-additivity of $\mathcal{L}^1$ on $\{E_h\}_{h\in\mathbb{N}}$ would then imply

$$1 = |(0, 1)| = \sum_{h\in\mathbb{N}} |E|,$$

against $|E| \in [0, \infty]$. Hence, $\mathcal{L}^1$ is not σ-additive on $\mathcal{P}(\mathbb{R})$. As we are going to prove in Section 2.1, $\mathcal{L}^1$ is, however, σ-additive on a large family of subsets of $\mathbb{R}^n$. A first step towards this kind of result is the following theorem, which provides, given outer measure μ, a natural domain of σ-additivity for μ.

Theorem 1.4 (Carathéodory's theorem) *If μ is an outer measure on $\mathbb{R}^n$, and $\mathcal{M}(\mu)$ is the family of those $E \subset \mathbb{R}^n$ such that*

$$\mu(F) = \mu(E \cap F) + \mu(F \setminus E)\,, \qquad \forall F \subset \mathbb{R}^n\,, \tag{1.5}$$

then $\mathcal{M}(\mu)$ is a σ-algebra, and μ is a measure on $\mathcal{M}(\mu)$.

Remark 1.5 We recall that $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^n)$ is a **σ-algebra** on $\mathbb{R}^n$ if $E \in \mathcal{M}$ implies $\mathbb{R}^n \setminus E \in \mathcal{M}$, $\{E_h\}_{h\in\mathbb{N}} \subset \mathcal{M}$ implies $\bigcup_{h\in\mathbb{N}} E_h \in \mathcal{M}$, and $\mathbb{R}^n \in \mathcal{M}$. If $\mathcal{M}$ is a σ-algebra, then a set function $\mu\colon \mathcal{M} \to [0,\infty]$ is a **measure** on $\mathcal{M}$ if $\mu(\emptyset) = 0$ and μ is σ-additive on $\mathcal{M}$.

Remark 1.6 A set E belongs to $\mathcal{M}(\mu)$ if it can be used to divide *any test set* $F \subset \mathbb{R}^n$ into two parts on which μ is additive. Notice that, by σ-subadditivity of μ, $E \in \mathcal{M}(\mu)$ if and only if

$$\mu(F) \ge \mu(F \setminus E) + \mu(F \cap E)\,, \qquad \forall F \subset \mathbb{R}^n \text{ s.t. } \mu(F) < \infty\,. \tag{1.6}$$

Elements of $\mathcal{M}(\mu)$ are called **μ-measurable sets**.

Proof *Step one:* We prove that $\mathcal{M}(\mu)$ is a σ-algebra. Clearly, $\emptyset \in \mathcal{M}(\mu)$ and $E \in \mathcal{M}(\mu)$ implies $\mathbb{R}^n \setminus E \in \mathcal{M}(\mu)$. We now let $\{E_h\}_{h\in\mathbb{N}} \subset \mathcal{M}(\mu)$, set $E = \bigcup_{h\in\mathbb{N}} E_h$, and prove that $E \in \mathcal{M}(\mu)$. Given $F \subset \mathbb{R}^n$, as $E_0 \in \mathcal{M}(\mu)$, we have

$$\mu(F) = \mu(F \setminus E_0) + \mu(F \cap E_0)\,.$$

As $E_1 \in \mathcal{M}(\mu)$ we also have

$$\begin{aligned}\mu(F \setminus E_0) &= \mu\Big((F \setminus E_0) \setminus E_1\Big) + \mu\Big((F \setminus E_0) \cap E_1\Big)\\ &= \mu\Big(F \setminus (E_0 \cup E_1)\Big) + \mu\Big((F \setminus E_0) \cap E_1\Big),\end{aligned}$$

and thus $\mu(F) = \mu(F \setminus (E_0 \cup E_1)) + \mu((F \setminus E_0) \cap E_1) + \mu(F \cap E_0)$. By induction,

$$\mu(F) = \mu\Big(F \setminus \bigcup_{h=0}^{k} E_h\Big) + \sum_{h=0}^{k} \mu\Big(\Big(F \setminus \bigcup_{j=0}^{h-1} E_j\Big) \cap E_h\Big), \tag{1.7}$$

for every $k \in \mathbb{N}$, $k \ge 1$. Since $F \setminus E \subset F \setminus \bigcup_{h=0}^{k} E_h$, by monotonicity we find

$$\mu(F) \ge \mu(F \setminus E) + \sum_{h=0}^{k} \mu\Big(\Big(F \setminus \bigcup_{j=0}^{h-1} E_j\Big) \cap E_h\Big).$$

Letting first $k \to \infty$, and then using σ-subadditivity, we find $E \in \mathcal{M}(\mu)$, as

$$\mu(F) \ge \mu(F \setminus E) + \sum_{h\in\mathbb{N}} \mu\Big(\Big(F \setminus \bigcup_{j=0}^{h-1} E_j\Big) \cap E_h\Big) \tag{1.8}$$

$$\ge \mu(F \setminus E) + \mu\Big(\bigcup_{h\in\mathbb{N}} \Big(F \setminus \bigcup_{j=0}^{h-1} E_j\Big) \cap E_h\Big) = \mu(F \setminus E) + \mu(F \cap E)\,.$$

Step two: We show that μ is σ-additive on $\mathcal{M}(\mu)$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a disjoint sequence in $\mathcal{M}(\mu)$. Setting $F = E = \bigcup_{h\in\mathbb{N}} E_h$ in (1.8), we find that $\mu(E) \ge \sum_{h\in\mathbb{N}} \mu(E_h)$. As μ is σ-subadditive, we conclude the proof of the theorem. □

Exercise 1.7 If μ and ν are outer measures on $\mathbb{R}^n$, then $\mu + \nu$ is an outer measure on $\mathbb{R}^n$, with $\mathcal{M}(\mu) \cap \mathcal{M}(\nu) \subset \mathcal{M}(\mu + \nu)$.

Exercise 1.8 If μ is an outer measure on $\mathbb{R}^n$ and $\{E_h\}_{h\in\mathbb{N}} \subset \mathcal{M}(\mu)$, then

$$E_h \subset E_{h+1}\,, \quad \forall h \in \mathbb{N} \qquad \Rightarrow \qquad \mu\Big(\bigcup_{h\in\mathbb{N}} E_h\Big) = \lim_{h\to\infty} \mu(E_h)\,,$$

$$\begin{cases} E_{h+1} \subset E_h\,, & \forall h \in \mathbb{N} \\ \mu(E_1) < \infty\,, \end{cases} \qquad \Rightarrow \qquad \mu\Big(\bigcap_{h\in\mathbb{N}} E_h\Big) = \lim_{h\to\infty} \mu(E_h)\,.$$

1.3 Measure Theory and integration

By Theorem 1.4, every outer measure on $\mathbb{R}^n$ can be seen as a measure on a σ-algebra on $\mathbb{R}^n$. In this way, various classical results from Measure Theory are immediately recovered in the context of outer measures. For the sake of clarity, in this chapter we gather those definitions and statements that will be used in the rest of the book. Let μ be a measure on the σ-algebra $\mathcal{M}$ on $\mathbb{R}^n$ (if μ is an outer measure on $\mathbb{R}^n$, then we take by convention $\mathcal{M} = \mathcal{M}(\mu)$). A function $u\colon E \to [-\infty, \infty]$ is a **μ-measurable function on $\mathbb{R}^n$** if its domain E covers μ-almost all of $\mathbb{R}^n$, that is $\mu(\mathbb{R}^n \setminus E) = 0$, and if, for every $t \in \mathbb{R}$, the super-level sets

$$\{u > t\} = \Big\{x \in E : u(x) > t\Big\}$$

belong to $\mathcal{M}$. We say that u is a **μ-simple function on $\mathbb{R}^n$** if u is μ-measurable and the image of u is countable. For a non-negative, μ-simple function u, the **integral of u with respect to μ** is defined in $[0, \infty]$ as the series

$$\int_{\mathbb{R}^n} u \,\mathrm{d}\mu = \sum_{t\in u(\mathbb{R}^n)} t\,\mu\big(\{u = t\}\big)\,,$$

with the convention that $0 \cdot \infty = 0$. When u is μ-simple, and either $\int_{\mathbb{R}^n} u^+ \, d\mu$ or $\int_{\mathbb{R}^n} u^- \, d\mu$ is finite (here, $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$), we say that u is a **μ-integrable simple function**, and set

$$\int_{\mathbb{R}^n} u \, d\mu = \int_{\mathbb{R}^n} u^+ \, d\mu - \int_{\mathbb{R}^n} u^- \, d\mu \, .$$

The **upper and lower integrals with respect to** μ of a function u whose domain covers μ-almost all of $\mathbb{R}^n$, and which takes values in $[-\infty, \infty]$, are

$$\int^*_{\mathbb{R}^n} u \, d\mu = \inf \left\{ \int_{\mathbb{R}^n} v \colon v \ge u \ \mu\text{-a.e. on } \mathbb{R}^n \right\},$$
$$\int_{*\mathbb{R}^n} u \, d\mu = \sup \left\{ \int_{\mathbb{R}^n} v \colon v \le u \ \mu\text{-a.e. on } \mathbb{R}^n \right\},$$

where v ranges over the family of μ-integrable simple functions on $\mathbb{R}^n$. If u is μ-measurable and its upper and lower integrals coincide, then we say that u is a **μ-integrable function**, and this common value is called the **integral of** u **with respect to** μ, denoted by $\int_{\mathbb{R}^n} u \, d\mu$. The following example suggests that μ-integrable functions define a large subfamily of μ-measurable functions.

Example 1.9 If u is μ-measurable on $\mathbb{R}^n$ and $u \ge 0$ μ-a.e. on $\mathbb{R}^n$, then u is μ-integrable. Indeed, if $\mu(\{u = \infty\}) > 0$, then for every $t > 0$ we have

$$\int_{*\mathbb{R}^n} u \, d\mu \ge t \, \mu\Big(\{u = \infty\}\Big),$$

so that, in particular, u is μ-integrable with $\int_{\mathbb{R}^n} u \, d\mu = \infty$. If, instead, $u(x) < \infty$ for μ-a.e. $x \in \mathbb{R}^n$, then given $t > 1$ we may construct a partition $\{E_h\}_{h \in \mathbb{Z}}$ of μ-almost all of $\mathbb{R}^n$ by setting $E_h = \{t^h \le u < t^{h+1}\}$, $h \in \mathbb{Z}$. By looking at the μ-simple functions $\sum_{h \in \mathbb{Z}} t^h 1_{E_h}$ and $\sum_{h \in \mathbb{Z}} t^{h+1} 1_{E_h}$, we thus conclude that

$$\int^*_{\mathbb{R}^n} u \, d\mu \le t \int_{*\mathbb{R}^n} u \, d\mu \, , \qquad \forall t > 1 \, .$$

Finally, u is a **locally μ-summable function**, or $u \in L^1_{\text{loc}}(\mathbb{R}^n, \mu)$, if it is μ-measurable and $\int_K |u| \, d\mu < \infty$ for every compact set $K \subset \mathbb{R}^n$; it is **μ-summable**, $u \in L^1(\mathbb{R}^n, \mu)$, if $\int_{\mathbb{R}^n} |u| \, d\mu < \infty$. The L^p-spaces $L^p(\mathbb{R}^n, \mu)$ and $L^p_{\text{loc}}(\mathbb{R}^n, \mu)$, $1 < p \le \infty$, are defined as usual. We shall also set for brevity $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n, \mathcal{L}^n)$.

Theorem (Monotone convergence theorem) *If $\{u_h\}_{h \in \mathbb{N}}$ is a sequence of μ-measurable functions $u_h \colon \mathbb{R}^n \to [0, \infty]$ such that $u_h \le u_{h+1}$ μ-a.e. on $\mathbb{R}^n$, then*

$$\lim_{h \to \infty} \int_{\mathbb{R}^n} u_h \, d\mu = \int_{\mathbb{R}^n} \sup_{h \in \mathbb{N}} u_h \, d\mu \, .$$