INTRODUCTION TO VASSILIEV KNOT INVARIANTS

With hundreds of worked examples, exercises and illustrations, this detailed exposition of the theory of Vassiliev knot invariants opens the field to students with little or no knowledge in this area. It also serves as a guide to more advanced material.

The book begins with a basic and informal introduction to knot theory, giving many examples of knot invariants before the class of Vassiliev invariants is introduced. This is followed by a detailed study of the algebras of Jacobi diagrams and 3-graphs, and the construction of functions on these algebras via Lie algebras. The authors then describe two constructions of a universal invariant with values in the algebra of Jacobi diagrams: via iterated integrals and via the Drinfeld associator, and extend the theory to framed knots. Various other topics are then discussed, such as Gauss diagram formulae, the diagrammatic version of the Duflo isomorphism, connection to the theory of nilpotent groups and more. The book ends with Vassiliev's original construction.

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To the memory of V. I. Arnold

Contents

Preface

This book is a detailed introduction to the theory of finite type (Vassiliev) knot invariants, with a stress on its combinatorial aspects. It is intended to serve both as a textbook for readers with no or little background in this area, and as a guide to some of the more advanced material. Our aim is to lead the reader to understanding by means of pictures and calculations, and for this reason we often prefer to convey the idea of the proof on an instructive example rather than give a complete argument. While we have made an effort to make the text reasonably self-contained, an advanced reader is sometimes referred to the original papers for the technical details of the proofs.

Historical remarks

The notion of a finite type knot invariant was introduced by Victor Vassiliev (Moscow) in the end of the 1980s and first appeared in print in his paper (1990a). Vassiliev, at the time, was not specifically interested in low-dimensional topology. His main concern was the general theory of discriminants in the spaces of smooth maps, and his description of the space of knots was just one, though the most spectacular, application of a machinery that worked in many seemingly unrelated contexts. It was V. I. Arnold (1992) who understood the importance of finite type invariants, coined the name "Vassiliev invariants" and popularized the concept; since that time, the term "Vassiliev invariants" has become standard.

A different perspective on the finite type invariants was developed by Mikhail Goussarov (St. Petersburg). His notion of n-equivalence, which first appeared in print in Goussarov (1993), turned out to be useful in different situations, for example, in the study of the finite type invariants of 3-manifolds.[1] Nowadays some people use the expression "Vassiliev–Goussarov invariants" for the finite type invariants.

[1] Goussarov cites Vassiliev's works in his earliest paper (1991). Nevertheless, according to O. Viro, Goussarov first mentioned finite type invariants in a talk at the Leningrad topological seminar as early as in 1987.

Vassiliev's definition of finite type invariants is based on the observation that knots form a topological space and the knot invariants can be thought of as the locally constant functions on this space. Indeed, the space of knots is an open subspace of the space M of all smooth maps from S^1 to $\mathbb{R}^3$; its complement is the so-called discriminant Σ which consists of all maps that fail to be embeddings. Two knots are isotopic if and only if they can be connected in M by a path that does not cross Σ.

Using simplicial resolutions, Vassiliev constructs a spectral sequence for the homology of Σ. After applying the Alexander duality, this spectral sequence produces cohomology classes for the space of knots $M - \Sigma$; in dimension zero these are precisely the Vassiliev knot invariants.

Vassiliev's approach, which is technically rather demanding, was simplified by J. Birman and X.-S. Lin (1993). They explained the relation between the Jones polynomial and finite type invariants[2] and emphasized the role of the algebra of chord diagrams. M. Kontsevich (1993) showed that the study of real-valued Vassiliev invariants can, in fact, be reduced entirely to the combinatorics of chord diagrams. His proof used an analytic tool (the Kontsevich integral) which is, essentially, a power series encoding all the finite type invariants of a knot. Kontsevich also defined a coproduct on the algebra of chord diagrams which turns it into a Hopf algebra.

D. Bar-Natan was the first to give a comprehensive treatment of Vassiliev knot and link invariants. In his preprint (1991a) and PhD thesis (1991b) he found the relationship between finite type invariants and the topological quantum field theory developed by his thesis advisor E. Witten (1989, 1995). Bar-Natan's paper (1995a) (whose preprint edition appeared in 1992) is still the most authoritative source on the fundamentals of the theory of Vassiliev invariants. About the same time, T. Le and J. Murakami (1996a), relying on V. Drinfeld's work (1989, 1990), proved the rationality of the Kontsevich integral.

Among further developments in the area of finite type knot invariants, let us mention:

- The existence of non-Lie-algebraic weight systems (Vogel 1997, Lieberum 1999) and an interpretation of all weight systems as Lie algebraic weight systems in a suitable category (Hinich and Vaintrob 2002);
- J. Kneissler's analysis (2000, 2001a, 2001b) of the structure of the algebra Λ introduced by P. Vogel (1997);
- The proof by Goussarov (1998b) that Vassiliev invariants are polynomials in the gleams for a fixed Turaev shadow;

[2] independently from Goussarov, who was the first to discover this relation in Goussarov (1991).

- Gauss diagram formulae of M. Polyak and O. Viro (1994) and the proof by M. Goussarov (1998a) that all finite type invariants can be expressed by such formulae;
- Habiro's theory of claspers (Habiro 2000) (see also Goussarov 2001);
- V. Vassiliev's papers (2001, 2005) where a general technique for deriving combinatorial formulae for cohomology classes in the complements to discriminants, and in particular, for finite type invariants, is proposed;
- Explicit formulae for the Kontsevich integral of some knots and links (Bar-Natan, Le and Thurston 2003; Bar-Natan and Lawrence 2004; Rozansky 2003; Kricker 2000; Marché 2004; Garoufalidis and Kricker 2004);
- The interpretation of the Vassiliev spectral sequence in terms of the Hochschild homology of the Poisson operad by V. Turchin (Tourtchine 2004);
- The alternative approaches to the topology of the space of knots via configuration spaces and the Goodwillie calculus (Sinha 2009).

One serious omission in this book is the connection between the Vassiliev invariants and the Chern–Simons theory. This connection motivates much of the interest in finite type invariants and gives a better understanding of the nature of the Kontsevich integral. Moreover, it suggests another form of the universal Vassiliev invariant, namely, the configuration space integral. There are many texts that explain this connection with great clarity; the reader may start, for instance, with Labastida (1999), Sawon (2006) or Polyak (2005). The original paper of Witten (1989) has not lost its relevance and, while it does not deal directly with the Vassiliev invariants (which were not defined at the time), it still is one of the indispensable references.

An important source of information on finite type invariants is the online *Bibliography of Vassiliev invariants* started by D. Bar-Natan and currently located at

`http://www.pdmi.ras.ru/~duzhin/VasBib/`

On January 19, 2011 it contained 641 items and this number is increasing. The study of finite type invariants is ongoing. However, notwithstanding all efforts, the most important question put forward in 1990:

Is it true that Vassiliev invariants distinguish knots?

is still open. At the moment it is not even known whether the Vassiliev invariants can detect knot orientation. A number of open problems related to finite type invariants are listed in Ohtsuki (2004b).

Prerequisites

We assume that the reader has a basic knowledge of calculus on manifolds (vector fields, differential forms, Stokes' theorem), general algebra (groups, rings, modules, Lie algebras, fundamentals of homological algebra), linear algebra (vector spaces, linear operators, tensor algebra, elementary facts about representations) and topology (topological spaces, homotopy, homology, Euler characteristic). Some of this and more advanced algebraic material (bialgebras, free algebras, universal enveloping algebras etc.), which is of primary importance in this book, can be found in the appendix at the end of the book. No knowledge of knot theory is presupposed, although it may be useful.

Contents

The book consists of fifteen chapters, which can logically be divided into four parts.

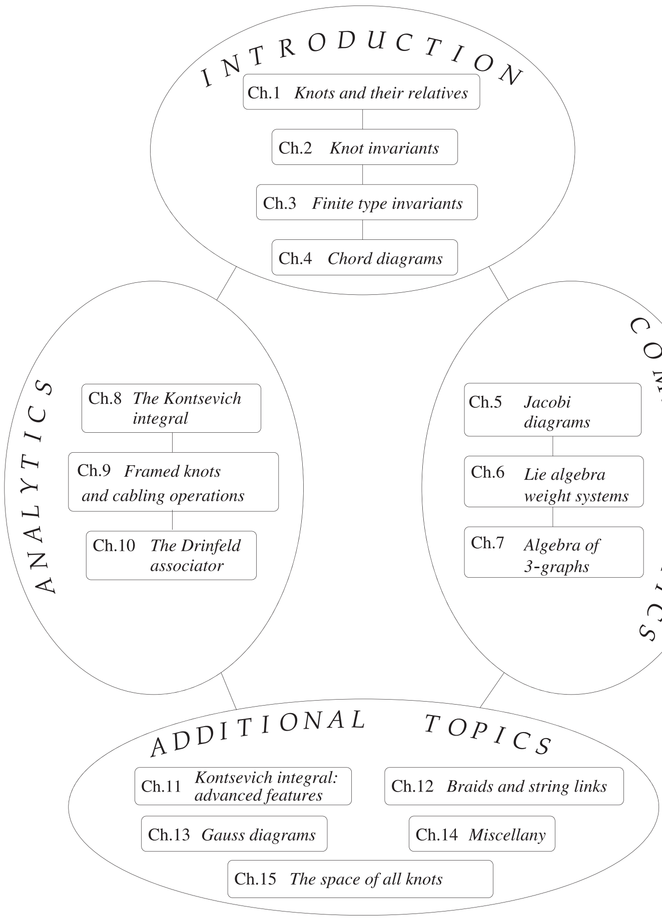
The first part, Chapters 1–4, opens with a short introduction into the theory of knots and their classical polynomial invariants and closes with the definition of Vassiliev invariants.

In Chapters 5–7, we systematically study the graded Hopf algebra naturally associated with the filtered space of Vassiliev invariants, which appears in three different guises: as the algebra of framed chord diagrams $\mathscr{A}$, as the algebra of closed Jacobi diagrams $\mathscr{C}$ and as the algebra of open Jacobi diagrams $\mathscr{B}$. After that, we study the auxiliary algebra Γ generated by regular trivalent graphs and closely related to the algebras $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ as well as to Vogel's algebra Λ. In Chapter 7 we discuss the weight systems defined by Lie algebras, both universal and depending on a chosen representation.

Chapters 8–10 are dedicated to a detailed exposition of the Kontsevich integral; it contains the proof of the main theorem of the theory of Vassiliev knot invariants that reduces their study to combinatorics of chord diagrams and related algebras. Chapters 8 and 9 treat the Kontsevich integral from the analytic point of view. Chapter 10 is dedicated to the Drinfeld associator and the combinatorial construction of the Kontsevich integral.

The last part of the book, Chapters 11–15, is devoted to various topics left out in the previous exposition. Chapter 11 contains some additional material on the Kontsevich integral: the wheels formula, the Rozansky rationality conjecture etc. This is followed in Chapters 12–14 by discussions of the Vassiliev invariants for braids, Gauss diagram formulae, the Melvin–Morton conjecture, the Goussarov–Habiro theory, the size of the space of Vassiliev invariants, etc. The book closes with a description of Vassiliev's original construction for the finite type invariants.

Chapter dependence



The book is intended to be a textbook, so we have included many exercises. Some exercises are embedded in the text; the others appear in a separate section at the end of each chapter. Open problems are marked with an asterisk.

Acknowledgements

The work of the first two authors on this book actually began in August 1992, when our colleague Inna Scherbak returned to Pereslavl-Zalessky from the First European Mathematical Congress in Paris and brought a photocopy of Arnold's lecture notes about the newborn theory of Vassiliev knot invariants. We spent

several months filling our waste-paper baskets with pictures of chord diagrams, before the first joint article (Chmutov and Duzhin 1994) was ready.

In the preparation of the present text, we have extensively used our papers (joint, single-authored and with other coauthors; see references) and in particular, lecture notes of the course "Vassiliev invariants and combinatorial structures" that one of us (S. D.) delivered at the Graduate School of Mathematics, University of Tokyo, in Spring 1999. It is our pleasure to thank V. I. Arnold, D. Bar-Natan, J. Birman, C. De Concini, O. Dasbach, A. Durfee, F. Duzhin, V. Goryunov, O. Karpenkov, T. Kerler, T. Kohno, S. Lando, M. Polyak, I. Scherbak, A. Sossinsky, V. Turchin, A. Vaintrob, A. Varchenko, V. Vassiliev and S. Willerton for many useful comments concerning the subjects touched upon in the book. We are likewise indebted to the anonymous referees whose criticism and suggestions helped us to improve the text.

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