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978-1-107-02083-2 — Introduction to Vassiliev Knot Invariants
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Knots and their relatives

This book is about knots. It is, however, hardly possible to speak about knots without mentioning other one-dimensional topological objects embedded into the three-dimensional space. Therefore, in this introductory chapter we give basic definitions and constructions pertaining to knots and their relatives: links, braids and tangles.

The table of knots provided in Table 1.1 will be used throughout the book as a source of examples and exercises.

1.1 Definitions and examples

1.1.1 Knots

A *knot* is a closed non-self-intersecting curve in 3-space. In this book, we shall mainly study smooth oriented knots. A precise definition can be given as follows.

Definition 1.1. A *parametrized knot* is an embedding of the circle S^1 into the Euclidean space $\mathbb{R}^3$.

Recall that an *embedding* is a smooth map which is injective and whose differential is nowhere zero. In our case, the non-vanishing of the differential means that the tangent vector to the curve is non-zero. In the above definition and everywhere in the sequel, the word *smooth* means *infinitely differentiable*.

A choice of an orientation for the parametrizing circle

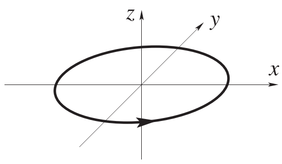
$$S^1 = \{(\cos t, \sin t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^2$$

gives an orientation to all the knots simultaneously. We shall always assume that S^1 is oriented counterclockwise. We shall also fix an orientation of the 3-space; each time we pick a basis for $\mathbb{R}^3$ we shall assume that it is consistent with the orientation.

If coordinates x, y, z are chosen in $\mathbb{R}^3$, a knot can be given by three smooth periodic functions of one variable $x(t)$, $y(t)$, $z(t)$.

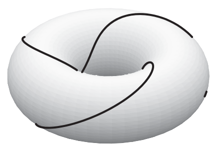
Example 1.2. The simplest knot is represented by a plane circle:

$$\begin{aligned} x &= \cos t, \\ y &= \sin t, \\ z &= 0. \end{aligned}$$



Example 1.3. The curve that goes 3 times around and 2 times across a standard torus in $\mathbb{R}^3$ is called the *(left) trefoil knot*, or the (2, 3)-*torus knot*:

$$\begin{aligned} x &= (2 + \cos 3t)\cos 2t, \\ y &= (2 + \cos 3t)\sin 2t, \\ z &= \sin 3t. \end{aligned}$$



Exercise. Give the definition of a (p, q)-torus knot. What are the appropriate values of p and q for this definition?

It will be convenient to identify knots that only differ by a change of a parametrization. An *oriented knot* is an equivalence class of parametrized knots under orientation-preserving diffeomorphisms of the parametrizing circle. Allowing *all* diffeomorphisms of S^1 in this definition, we obtain *unoriented knots*. Alternatively, an unoriented knot can be defined as the *image* of an embedding of S^1 into $\mathbb{R}^3$; an oriented knot is then an image of such an embedding together with the choice of one of the two possible directions on it.

We shall distinguish oriented/unoriented knots from parametrized knots in the notation: oriented and unoriented knots usually will be denoted by capital letters, while for the individual embeddings lowercase letters will be used. As a rule, the word “knot” will mean “oriented knot,” unless it is clear from the context that we deal with unoriented knots, or consider a specific choice of parametrization.

1.1.2 Isotopy

The study of parametrized knots falls within the scope of differential geometry. The *topological* study of knots requires an equivalence relation which would not only discard the specific choice of parametrization, but also model the physical transformations of a closed piece of rope in space.

By a *smooth family of maps*, or a *map smoothly depending on a parameter*, we understand a smooth map $F : S^1 \times I \to \mathbb{R}^3$, where $I \subset \mathbb{R}$ is an interval. Assigning a fixed value a to the second argument of F, we get a map $f_a : S^1 \to \mathbb{R}^3$.

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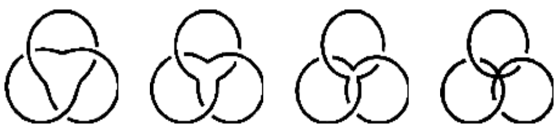
Definition 1.4. A smooth isotopy of a knot $f : S^1 \to \mathbb{R}^3$, is a smooth family of knots f_u, with u a real parameter, such that for some value $u = a$ we have $f_a = f$.

For example, the formulae

$$\begin{aligned} x &= (u + \cos 3t) \cos 2t, \\ y &= (u + \cos 3t) \sin 2t, \\ z &= \sin 3t, \end{aligned}$$

where $u \in (1, +\infty)$, represent a smooth isotopy of the trefoil knot which corresponds to $u = 2$. In the pictures below, the space curves are shown by their projection to the (x, y) plane:

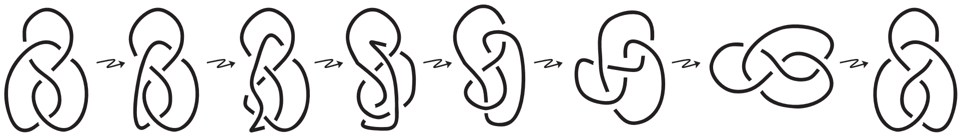


$u = 2$ $u = 1.5$ $u = 1.2$ $u = 1$

For any $u > 1$ the resulting curve is smooth and has no self-intersections, but as soon as the value $u = 1$ is reached we get a singular curve with three coinciding cusps[1] corresponding to the values $t = \pi/3$, $t = \pi$ and $t = 5\pi/3$. This curve is not a knot.

Definition 1.5. Two parametrized knots are said to be *isotopic* if one can be transformed into another by means of a smooth isotopy. Two oriented knots are isotopic if they represent the classes of isotopic parametrized knots; the same definition is valid for unoriented knots.

Example 1.6. This picture shows an isotopy of the figure eight knot into its mirror image:



There are other notions of knot equivalence, namely, *ambient equivalence* and *ambient isotopy*, which, for smooth knots, are the same thing as isotopy. Here are the definitions. A proof that they are equivalent to our definition of isotopy can be found in Burde and Zieschang (2003).

[1] A *cusp* of a spatial curve is a point where the curve can be represented as $x = s^2$, $y = s^3$, $z = 0$ in some local coordinates.

Definition 1.7. Two parametrized knots, f and g, are *ambient equivalent* if there is a commutative diagram

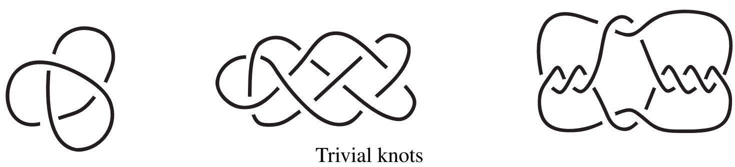
$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathbb{R}^3 \arrow[d, "\psi"] \\ S^1 \arrow[r, "g"] & \mathbb{R}^3 \end{tikzcd}$$

where φ and ψ are orientation-preserving diffeomorphisms of the circle and the 3-space, respectively.

Definition 1.8. Two parametrized knots, f and g, are *ambient isotopic* if there is a smooth family of diffeomorphisms of the 3-space $\psi_t : \mathbb{R}^3 \to \mathbb{R}^3$ with $\psi_0 = \mathrm{id}$ and $\psi_1 \circ f = g$.

Definition 1.9. A knot, equivalent to the plane circle on page 2 is referred to as a *trivial knot*, or an *unknot*.

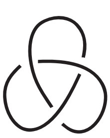
Sometimes, it is not immediately clear from a diagram of a trivial knot that it is indeed trivial:



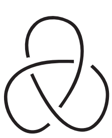
Trivial knots

There are algorithmic procedures to detect whether a given knot diagram represents an unknot. One of them, based on W. Thurston's ideas, is implemented in J. Weeks' computer program `SnapPea`; see Weeks (2010); another algorithm, due to I. Dynnikov, is described in Dynnikov (2006).

Here are several other examples of knots.

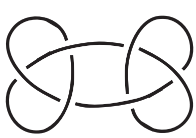


Left trefoil

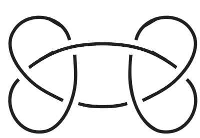


Right trefoil

Figure 8 knot



Granny knot

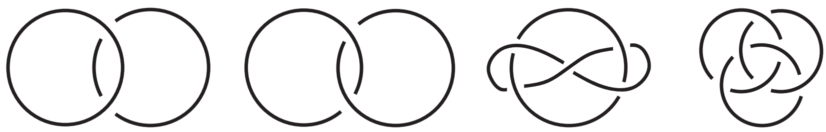


Square knot

1.1.3 Links

Knots are a special case of links.

Definition 1.10. A *link* is a smooth embedding $S^1 \sqcup \cdots \sqcup S^1 \to \mathbb{R}^3$, where $S^1 \sqcup \cdots \sqcup S^1$ is the disjoint union of several circles.



Trivial 2-component link Hopf link Whitehead link Borromean rings

Equivalence of links is defined in the same way as for knots – with the exception that now one may choose whether or not to distinguish between the components of a link and thus speak about the equivalence of links with *numbered* or *unnumbered* components.

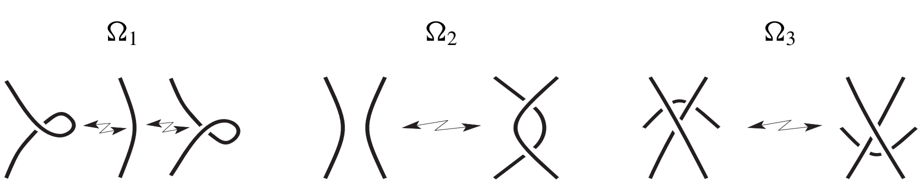
In the future, we shall often say "knot (link)" instead of "equivalence class," or "topological type of knots (links)."

1.2 Plane knot diagrams

1.2.1 Knot diagrams

Knots are best represented graphically by means of *knot diagrams*. A knot diagram is a plane curve whose only singularities are transversal double points (crossings), together with the choice of one branch of the curve at each crossing. The chosen branch is called an *overcrossing*; the other branch is referred to as an *undercrossing*. A knot diagram is thought of as a projection of a knot along some "vertical" direction; overcrossings and undercrossings indicate which branch is "higher" and which is "lower." To indicate the orientation, an arrow is added to the knot diagram.

Theorem 1.11 (Reidemeister 1948, proofs can be found in Prasolov and Sossinsky 1997; Burde and Zieschang 2003; and Murasugi 1996). *Two unoriented knots, K_1 and K_2, are equivalent if and only if a diagram of K_1 can be transformed into a diagram of K_2 by a sequence of isotopies of the plane and local moves of the following three types:*



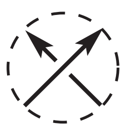
Reidemeister moves

To adjust the assertion of this theorem to the oriented case, each of the three Reidemeister moves has to be equipped with orientations in all possible ways. Smaller sufficient sets of oriented moves exist; one such set will be given later in terms of Gauss diagrams (see Section 1.7.3).

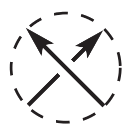
Exercise. Determine the sequence of Reidemeister moves that relates the two diagrams of the trefoil knot below:

1.2.2 Local writhe

Crossing points on a diagram come in two species, positive and negative:



Positive crossing



Negative crossing

Although this sign is defined in terms of the knot orientation, it is easy to check that it does not change if the orientation is reversed. For *links* with more than one component, the choice of orientation is essential.

The *local writhe* of a crossing is defined as $+1$ or -1 for positive or negative points, respectively. The *writhe* (or *total writhe*) of a diagram is the sum of the writhes of all crossing points, or, equivalently, the difference between the number of positive and negative crossings. Of course, the same knot may be represented by diagrams with different total writhes. In Chapter 2 we shall see how the writhe can be used to produce knot invariants.

1.2.3 Alternating knots

A knot diagram is called *alternating* if its overcrossings and undercrossing alternate as we travel along the knot. A knot is called *alternating* if it has an alternating diagram. A knot diagram is called *reducible* if it becomes disconnected after the removal of a small neighbourhood of some crossing.

The number of crossings in a reducible diagram can be decreased by a move shown in the picture:

reducible diagram

reduction

A diagram which is not reducible is called *reduced*. As there is no immediate way to simplify a reduced diagram, the following conjecture naturally arises (Tait, 1898).

The Tait conjecture. *A reduced alternating diagram has the minimal number of crossings among all diagrams of the given knot.*

This conjecture stood open for almost 100 years. It was proved only in 1986 (using the newly invented Jones polynomial) simultaneously and independently by L. Kauffman (1987b), K. Murasugi (1987) and M. Thistlethwaite (1987) (see Exercise 2.27).

1.3 Inverses and mirror images

Change of orientation (taking the inverse) and taking the mirror image are two basic operations on knots which are induced by orientation reversing smooth involutions on S^1 and $\mathbb{R}^3$ respectively. Every such involution on S^1 is conjugate to the reversal of the parametrization; on $\mathbb{R}^3$ it is conjugate to a reflection in a plane mirror.

Let K be a knot. Composing the parametrization reversal of S^1 with the map $f : S^1 \to \mathbb{R}^3$ representing K, we obtain the *inverse* K^* of K. The *mirror image* of K, denoted by $\overline{K}$, is a composition of the map $f : S^1 \to \mathbb{R}^3$ with a reflection in $\mathbb{R}^3$. Both change of orientation and taking the mirror image are involutions on the set of (equivalence classes of) knots. They generate a group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$; the symmetry properties of a knot K depend on the subgroup that leaves the knot invariant. The group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ has five (not necessarily proper) subgroups, which give rise to five symmetry classes of knots.

Definition 1.12. A knot is called:

- *invertible*, if $K^* = K$,
- *plus-amphicheiral*, if $\overline{K} = K$,
- *minus-amphicheiral*, if $\overline{K} = K^*$,
- *fully symmetric*, if $K = K^* = \overline{K} = \overline{K}^*$,
- *totally asymmetric*, if all knots K, K^*, $\overline{K}$, $\overline{K}^*$ are different.

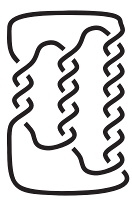
The word *amphicheiral* means either plus- or minus-amphicheiral. For invertible knots, this is the same. Amphicheiral and non-amphicheiral knots are also referred to as *achiral* and *chiral* knots, respectively.

The five symmetry classes of knots are summarized in the following table. The word “minimal” means “with the minimal number of crossings”; σ and τ denote the involutions of taking the mirror image and the inverse respectively. The notation for concrete knots in the last column will be explained in the next section.

Stabilizer	Orbit	Symmetry type	Min example
$\{1\}$	$\{K, \overline{K}, K^*, \overline{K}^*\}$	totally asymmetric	9_{32}, 9_{33}
$\{1, \sigma\}$	$\{K, K^*\}$	+amphicheiral, non-inv	12^a_{427}
$\{1, \tau\}$	$\{K, \overline{K}\}$	invertible, chiral	3_1
$\{1, \sigma\tau\}$	$\{K, K^*\}$	−amphicheiral, non-inv	8_{17}
$\{1, \sigma, \tau, \sigma\tau\}$	$\{K\}$	fully symmetric	4_1

Example 1.13. The trefoil knots are invertible, because the rotation through 180° around an axis in $\mathbb{R}^3$ changes the direction of the arrow on the knot.

The existence of non-invertible knots was first proved by H. Trotter (1964). The simplest instance of Trotter's theorem is a *pretzel knot with parameters* (3, 5, 7):



Among the knots with up to eight crossings (see Table 1.1) there is only one non-invertible knot: 8_{17}, which is, moreover, minus-amphicheiral. These facts were proved by A. Kawauchi (1979).

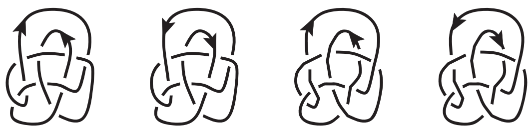
Example 1.14. The trefoil knots are not amphicheiral, hence the distinction between the left and the right trefoil. A proof of this fact, based on the calculation of the Jones polynomial, will be given in Section 2.4.

Remark 1.15. Knot tables only list knots up to taking inverses and mirror images. In particular, there is only one entry for the trefoil knots. Either of them is often referred to as *the trefoil*.

Example 1.16. The figure eight knot is amphicheiral. The isotopy between this knot and its mirror image is shown in Example 1.6.

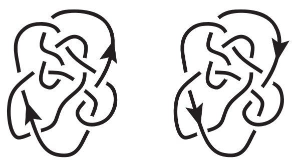
Among the 35 knots with up to eight crossings shown in Table 1.1, there are exactly seven amphicheiral knots: 4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{17}, 8_{18}, out of which 8_{17} is minus-amphicheiral, the rest, as they are invertible, are both plus- and minus-amphicheiral.

The simplest totally asymmetric knots appear in nine crossings, they are 9_{32} and 9_{33}. The following are all non-equivalent:



9_{33} $\quad 9^*_{33}$ $\quad \overline{9}_{33}$ $\quad \overline{9}^*_{33}$

Here is the simplest plus-amphicheiral non-invertible knot, together with its inverse:



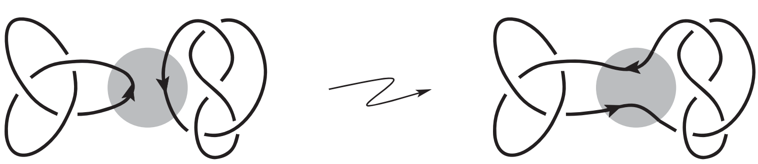
12^a_{427} $\quad 12^{a*}_{427}$

In practice, the easiest way to find the symmetry type of a given knot or link is by using the computer program Knotscape (Hoste and Thistlethwaite 1999), which can handle link diagrams with up to 49 crossings.

1.4 Knot tables

1.4.1 Connected sum

There is a natural way to fuse two knots into one: cut each of the two knots at some point, then connect the two pairs of loose ends. This must be done with some caution: first, by a smooth isotopy, both knots should be deformed so that for a certain plane projection they look as shown in the picture below on the left, then they should be changed inside the dashed disk as shown on the right:



The connected sum makes sense only for oriented knots. It is well-defined and commutative on the equivalence classes of knots. The connected sum of knots K_1 and K_2 is denoted by $K_1 \# K_2$.

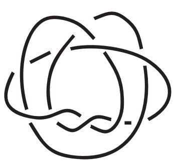
Definition 1.17. A knot is called *prime* if it cannot be represented as the connected sum of two nontrivial knots.

Each knot is a connected sum of prime knots, and this decomposition is unique (see Crowell and Fox (1963) for a proof). In particular, this means that a trivial knot cannot be decomposed into a sum of two nontrivial knots. Therefore, in order to classify all knots, it is enough to have a table of all prime knots.

1.4.2 Knot tables

Prime knots are tabulated according to the minimal number of crossings that their diagrams can have. Within each group of knots with the same crossing number, knots are numbered in some, usually rather arbitrary, way. In Table 1.1, we use the widely adopted numbering that goes back to the table compiled by Alexander and Briggs (1926/1927), then repeated (in an extended and modified way) by D. Rolfsen (1976). We also follow Rolfsen's conventions in the choice of the version of non-amphicheiral knots: for example, our 3_1 is the left, not the right, trefoil.

Rolfsen's table of knots, authoritative as it is, contained an error. It is the famous *Perko pair* (knots 10_{161} and 10_{162} in Rolfsen) – two equivalent knots that were thought to be different for 75 years since 1899:



The equivalence of these two knots was established in 1973 by K. A. Perko (Perko 1973), a lawyer from New York who studied mathematics at Princeton in 1960–1964 (Perko 2002) but later chose jurisprudence to be his profession.[2]

Complete tables of knots are currently known up to crossing number 16 (Hoste, Thistlethwaite and Weeks 1998). For knots with 11 through 16 crossings it is nowadays customary to use the numbering of Knotscape (Hoste and Thistlethwaite 1999) where the tables are built into the software. For each crossing number, Knotscape has a separate list of alternating and non-alternating knots. For example, the notation 12^a_{427} used in Section 1.3, refers to item number 427 in the list of alternating knots with 12 crossings.

1.5 Algebra of knots

Denote by $\mathscr{K}$ the set of the equivalence classes of knots. It forms a commutative monoid (semigroup with a unit) under the connected sum of knots, and therefore we can construct the monoid algebra $\mathbb{Z}\mathscr{K}$ of $\mathscr{K}$. By definition, elements of $\mathbb{Z}\mathscr{K}$ are formal finite linear combinations $\sum \lambda_i K_i$, $\lambda_i \in \mathbb{Z}$, $K_i \in \mathscr{K}$, the product

[2]The combination of a professional lawyer and an amateur mathematician in one person is not new in the history of mathematics (think of Pierre Fermat!).