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978-1-107-02040-5 - A Basic Course in Measure and Probability: Theory for Applications

Ross Leadbetter, Stamatis Cambanis and Vladas Pipiras

Frontmatter

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A Basic Course in Measure and Probability

Originating from the authors' own graduate course at the University of North Carolina, this material has been thoroughly tried and tested over many years, making the book perfect for a two-term course or for self-study. It provides a concise introduction that covers all of the measure theory and probability most useful for statisticians, including Lebesgue integration, limit theorems in probability, martingales, and some theory of stochastic processes. Readers can test their understanding of the material through the 300 exercises provided.

The book is especially useful for graduate students in statistics and related fields of application (biostatistics, econometrics, finance, meteorology, machine learning, etc.) who want to shore up their mathematical foundation. The authors establish common ground for students of varied interests, which will serve as a firm "take-off point" for them as they specialize in areas that exploit mathematical machinery.

ROSS LEADBETTER is Professor of Statistics and Operations Research at the University of North Carolina, Chapel Hill. His research involves stochastic process theory, point processes, particularly extreme value and risk theory for stationary sequences and processes, and applications to engineering, oceanography, and the environment.

STAMATIS CAMBANIS was a Professor at the University of North Carolina, Chapel Hill until his death in 1995, his research including fundamental contributions to stochastic process theory, and especially stable processes. He taught a wide range of statistics and probability courses and contributed very significantly to the development of the measure and probability instruction and the lecture notes on which this volume is based.

VLADAS PIPIRAS has been with the University of North Carolina, Chapel Hill since 2002, and a full Professor since 2012. His main research interests focus on stochastic processes exhibiting long-range dependence, multifractality and other scaling phenomena, as well as on stable, extreme value and other distributions possessing heavy tails. He has also worked on statistical inference questions for reduced-rank models with applications to econometrics, and sampling issues for finite point processes with applications to data traffic modeling in computer networks.

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Frontmatter

[More information](#)

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Frontmatter

[More information](#)

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Theory for Applications

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CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781107020405

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First published 2014

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

Leadbetter, Ross, author.

A basic course in measure and probability : theory for applications / Ross Leadbetter, Stamatis Cambanis, Vlasdas Pipiras.

pages cm

ISBN 978-1-107-02040-5 (hardback)

ISBN 978-1-107-65252-1 (paperback)

1. Measure theory. 2. Probabilities. I. Cambanis, Stamatis, 1943-1995, author. II. Pipiras, Vlasdas, author. III. Title.

QC20.7.M43L43 2013

515'.42-dc23 2013028841

ISBN 978-1-107-02040-5 Hardback

ISBN 978-1-107-65252-1 Paperback

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Ross Leadbetter, Stamatis Cambanis and Vlasos Pipiras

Frontmatter

[More information](#)

Contents

<i>Preface</i>	<i>page</i> ix
<i>Acknowledgements</i>	xiii
1 Point sets and certain classes of sets	1
1.1 Points, sets and classes	1
1.2 Notation and set operations	2
1.3 Elementary set equalities	5
1.4 Limits of sequences of sets	6
1.5 Indicator (characteristic) functions	7
1.6 Rings, semirings, and fields	8
1.7 Generated rings and fields	11
1.8 σ -rings, σ -fields and related classes	13
1.9 The real line – Borel sets	16
Exercises	18
2 Measures: general properties and extension	21
2.1 Set functions, measure	21
2.2 Properties of measures	23
2.3 Extension of measures, stage 1: from semiring to ring	27
2.4 Measures from outer measures	29
2.5 Extension theorem	31
2.6 Completion and approximation	34
2.7 Lebesgue measure	37
2.8 Lebesgue–Stieltjes measures	39
Exercises	41
3 Measurable functions and transformations	44
3.1 Measurable and measure spaces, extended Borel sets	44
3.2 Transformations and functions	45

Cambridge University Press

978-1-107-02040-5 - A Basic Course in Measure and Probability: Theory for Applications

Ross Leadbetter, Stamatis Cambanis and Vladas Pipiras

Frontmatter

[More information](#)

vi	<i>Contents</i>	
3.3	Measurable transformations and functions	47
3.4	Combining measurable functions	50
3.5	Simple functions	54
3.6	Measure spaces, “almost everywhere”	57
3.7	Measures induced by transformations	58
3.8	Borel and Lebesgue measurable functions	59
	Exercises	60
4	The integral	62
4.1	Integration of nonnegative simple functions	62
4.2	Integration of nonnegative measurable functions	63
4.3	Integrability	68
4.4	Properties of the integral	69
4.5	Convergence of integrals	73
4.6	Transformation of integrals	77
4.7	Real line applications	78
	Exercises	80
5	Absolute continuity and related topics	86
5.1	Signed and complex measures	86
5.2	Hahn and Jordan decompositions	87
5.3	Integral with respect to signed measures	92
5.4	Absolute continuity and singularity	94
5.5	Radon–Nikodym Theorem and the Lebesgue decomposition	96
5.6	Derivatives of measures	102
5.7	Real line applications	104
	Exercises	112
6	Convergence of measurable functions, L_p-spaces	118
6.1	Modes of pointwise convergence	118
6.2	Convergence in measure	120
6.3	Banach spaces	124
6.4	The spaces L_p	127
6.5	Modes of convergence – a summary	134
	Exercises	135
7	Product spaces	141
7.1	Measurability in Cartesian products	141
7.2	Mixtures of measures	143
7.3	Measure and integration on product spaces	146
7.4	Product measures and Fubini’s Theorem	149
7.5	Signed measures on product spaces	152
7.6	Real line applications	153
7.7	Finite-dimensional product spaces	155

Cambridge University Press

978-1-107-02040-5 - A Basic Course in Measure and Probability: Theory for Applications

Ross Leadbetter, Stamatis Cambanis and Vlasos Pipiras

Frontmatter

[More information](#)

<i>Contents</i>		vii
7.8	Lebesgue–Stieltjes measures on \mathbb{R}^n	158
7.9	The space $(\mathbb{R}^T, \mathcal{B}^T)$	163
7.10	Measures on \mathbb{R}^T , Kolmogorov’s Extension Theorem	167
	Exercises	170
8	Integrating complex functions, Fourier theory and related topics	177
8.1	Integration of complex functions	177
8.2	Fourier–Stieltjes, and Fourier Transforms in L_1	180
8.3	Inversion of Fourier–Stieltjes Transforms	182
8.4	“Local” inversion for Fourier Transforms	186
9	Foundations of probability	189
9.1	Probability space and random variables	189
9.2	Distribution function of a random variable	191
9.3	Random elements, vectors and joint distributions	195
9.4	Expectation and moments	199
9.5	Inequalities for moments and probabilities	200
9.6	Inverse functions and probability transforms	203
	Exercises	204
10	Independence	208
10.1	Independent events and classes	208
10.2	Independent random elements	211
10.3	Independent random variables	213
10.4	Addition of independent random variables	216
10.5	Borel–Cantelli Lemma and zero-one law	217
	Exercises	219
11	Convergence and related topics	223
11.1	Modes of probabilistic convergence	223
11.2	Convergence in distribution	227
11.3	Relationships between forms of convergence	235
11.4	Uniform integrability	238
11.5	Series of independent r.v.’s	241
11.6	Laws of large numbers	247
	Exercises	249
12	Characteristic functions and central limit theorems	254
12.1	Definition and simple properties	254
12.2	Characteristic function and moments	258
12.3	Inversion and uniqueness	261
12.4	Continuity theorem for characteristic functions	263
12.5	Some applications	265

Cambridge University Press

978-1-107-02040-5 - A Basic Course in Measure and Probability: Theory for Applications

Ross Leadbetter, Stamatis Cambanis and Vlasos Pipiras

Frontmatter

[More information](#)

viii

Contents

12.6	Array sums, Lindeberg–Feller Central Limit Theorem	268
12.7	Recognizing a c.f. – Bochner’s Theorem	271
12.8	Joint characteristic functions	277
	Exercises	280
13	Conditioning	285
13.1	Motivation	285
13.2	Conditional expectation given a σ -field	287
13.3	Conditional probability given a σ -field	291
13.4	Regular conditioning	293
13.5	Conditioning on the value of a r.v.	300
13.6	Regular conditional densities	303
13.7	Summary	305
	Exercises	306
14	Martingales	309
14.1	Definition and basic properties	309
14.2	Inequalities	314
14.3	Convergence	319
14.4	Centered sequences	325
14.5	Further applications	330
	Exercises	337
15	Basic structure of stochastic processes	340
15.1	Random functions and stochastic processes	340
15.2	Construction of the Wiener process in $\mathbb{R}^{[0,1]}$	343
15.3	Processes on special subspaces of \mathbb{R}^T	344
15.4	Conditions for continuity of sample functions	345
15.5	The Wiener process on C and Wiener measure	346
15.6	Point processes and random measures	347
15.7	A purely measure-theoretic framework for r.m.’s	348
15.8	Example: The sample point process	350
15.9	Random element representation of a r.m.	351
15.10	Mixtures of random measures	351
15.11	The general Poisson process	353
15.12	Special cases and extensions	354
	<i>References</i>	356
	<i>Index</i>	357

Cambridge University Press

978-1-107-02040-5 - A Basic Course in Measure and Probability: Theory for Applications

Ross Leadbetter, Stamatis Cambanis and Vlasos Pipiras

Frontmatter

[More information](#)

Preface

This work arises from lecture notes for a two semester basic course sequence in Measure and Probability Theory given for first year Statistics graduate students at the University of North Carolina, evolving through many generations of handwritten, typed, mimeographed, and finally LaTeX editions. Their focus is to provide basic course material, tailored to the background of our students, and influenced very much by their reactions and the changing emphases of the years. We see this as one side of an avowed department educational mission to provide solid and diverse basic course training common to all our students, who will later specialize in diverse areas from the very theoretical to the very applied.

The notes originated in the 1960's from a "Halmos style" measure theory course. As may be apparent (to those of sufficient age) the measure theory section has preserved that basic flavor with numerous obvious modernizations (beginning with the early use of the Sierpinski-type classes more suited than monotone class theorems for probabilistic applications), and exposition more tailored to the particular audience. Even the early "Halmos framework" of rings and σ -rings has been retained up to a point since these notions are useful in applications (e.g. point process theory) and their inclusion requires no significant further effort. Integration itself is discussed within the customary σ -field framework so the students have no difficulty in relating to other works.

Strong opinions abound as to how measure theory should be taught, or even if it should be taught: its existence was once described by a Danish statistical colleague as an "unfortunate historical accident" and by a local mathematician as an "unnatural way of approaching integration." In particular he felt that the Caratheodory extension "was not natural" since,

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Ross Leadbetter, Stamatis Cambanis and Vladas Pipiras

Frontmatter

[More information](#)

as he expressed it “If Caratheodory had not thought of it, I wouldn’t have either!”

Perhaps more threatening is the “bottom line” climate in some of today’s universities suggesting that training in measure-theoretic probability and statistical theory belongs to the past and should be deemphasized in favor of concentrated computational training for modern project-oriented activity. In this respect we can point with great pride to the many of our graduates making substantial statistical contributions in applications ascribable in (excuse us) “significant measure” to a solid theoretical component in their training. Moreover we ourselves see rather dramatic enrollment increases in our graduate probability courses from students in other disciplines in our own university and beyond, in fields such as financial mathematics with basic probability prerequisite. These (at least local) factors suggest a continuing role for both basic and more advanced course offerings, with the opportunity for innovative selection of special topics to be included.

Our viewpoint regarding presentation, much less single minded than some, is that we would teach (even name) this subject differently according to the particular audience needs. Based on the typical “advanced calculus” and “operational probability” backgrounds of our own students we prefer an essentially non-topological measure theory course followed by one in basic probability theory. For those of a more mathematical bent, the beautiful interplay between measure, topology (and algebra) can be studied at a later stage and is not a substantial part of our standard training mission for first year statistics graduate students. This organization has the incidental advantage that those who do further study have gained an understanding of which arguments (such as the central “ σ -ring game”) are measure theoretic in nature in contrast to being topological, or algebraic.

Our aim in the first semester is to provide a comprehensive account of general measure and integration theory. This we see as a quite well and naturally defined body of topics, generalizing much of standard real line Lebesgue integration theory to abstract spaces. Indeed a valuable by-product is that a student may automatically acquire an understanding of real line Lebesgue integration and its relationship to Riemann theory, made visible by a supply of exercises involving real line applications. We find it natural to first treat this body of (general measure) theory, giving advance glimpses from time to time of the probabilistic context. Some authors prefer the immediacy of probabilistic perspective attainable from a primary focus on probability in development *ab initio*, with extensions to general measures being indicated to the degree desired. This is primarily a

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Ross Leadbetter, Stamatis Cambanis and Vlasos Pipiras

Frontmatter

[More information](#)*Preface*

xi

question of purpose and taste with pros and cons. The only viewpoint we would strongly disagree with is that there exists a uniformly best didactic approach.

In the context of “measure theory” we view σ -finiteness as the “natural norm” for the statement of results, and finite measures as (albeit important) special cases. This, naturally, changes in the second part with primary focus on probability measures and more special resulting theory. In addition to the specialization of general measure theoretic results to yield the basic framework for probability theory there is, of course, an unlimited variety of results which may be explored in the purely probabilistic context and one may argue about which are truly central and a *sine qua non* for a one-semester treatment. There would probably be little disagreement with the topics we have included as being necessary and desirable knowledge, but they certainly cannot be regarded as sufficient for all students. Again our guiding principle has been to provide a course suited as common ground for our students of varied interests and serving as a “take-off point” for them as they specialize in areas ranging from applied statistics to stochastic analysis.

For a course one has to decide whether to emphasize basic ideas, details, or both. We have certainly attempted to strongly highlight the central ideas; if we have erred it is in the direction of including as complete details as possible, feeling that these should be seen at least once by the students. For example, detailed consideration of sets of measure zero, of possibly infinite function values and the specific identification of $XxYxZ$ with $(XxY)xZ$ are not necessarily issues of lasting emphasis in practice but we think it appropriate and desirable to deal with them carefully when introduced in a course. As will be clear, it has not been our intention to produce yet one more comprehensive book on this subject. Rather we have used the facilities of modern word processing as encouragement to give our lecture notes a better organized and repeatedly updated basic course form in the hope that they (and now this volume) will be the more useful to our own students, for whom they are designed, and to others who may share our educational perspectives.

Finally, it is with more than a twinge of sadness that this preface is written in the absence of coauthor Stamatis Cambanis, without whom the lecture notes would not have taken on any really comprehensive form. From the rough (mainly measure - theoretic) notes prepared by MRL in the 1960's, SC and MRL worked together in developing the notes from the mid-1970's as they taught the classes, until Stamatis' untimely death in 1995.

Cambridge University Press

978-1-107-02040-5 - A Basic Course in Measure and Probability: Theory for Applications

Ross Leadbetter, Stamatis Cambanis and Vlasos Pipiras

Frontmatter

[More information](#)

xii

Preface

Stamatis Cambanis was a wonderfully sensitive human being and friend, with unmatched concern to give help wherever and whatever the need. He was also The Master Craftsman in all that he did, his character echoing the words of Aristotle: “Είμαστε αυτό που πράττουμε επανειλημμένα. Έτσι, η τελειότητα δεν είναι πράξη αλλά συνήθεια.” (*We are what we repeatedly do. Excellence then is not an act but a habit.*)

M.R.L., V.P.

Acknowledgements

It is indeed hazardous to list acknowledgements in a work that has been used in developing form for almost half a century, and we apologize in advance for inevitable memory lapses that have caused omissions. It goes without saying that we are grateful to generations of questioning students, often indicating some lack of clarity of exposition in class or in the notes, and leading to needed revisions. Some have studied sections of special interest to them and not infrequently challenged details or phrasing of proofs – again leading to improvements in clarity. In particular Chihoon Lee undertook a quite unsolicited examination of the entire set of notes and pointed out many typographic and other blemishes at that time. Xuan Wang reviewed the entire manuscript in detail. We are especially grateful to Martin Heller who critically reviewed the entire set of book proofs and has prepared a solution set for many of the exercises.

Typing of original versions of the notes was creatively done by Peggy Ravitch and Harrison Williams, who grappled with the early mysteries of LaTeX, pioneered its use in the department, and constantly found imaginative ways to outwit its firm rules. Further residual typing was willingly done by Jiang Chen, James Wilson and Stefanos Kechagias, who also doubled as Greek linguistics advisor. It is a pleasure to record the encouragement and helpful comments of our colleague Amarjit Budhiraja who used the notes as supplementary material for his classes, and the repeated nagging of Climatologist Jerry Davis for publication as a book, as he used the notes as background in his research.

We are especially grateful to the Institute of Mathematical Statistics and the Editors of the IMS Lecture Note Series Anirban DasGupta and the inimitable Susan Murphy for their enthusiasm for production as a volume,

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978-1-107-02040-5 - A Basic Course in Measure and Probability: Theory for Applications

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Frontmatter

[More information](#)

and for the conversion of the entire manuscript from older LaTeX and hand corrected pdf files into the new format, through Mattson Publishing Company, the ever patient and gracious Geri Mattson, and the magical group VTeX. In particular we thank IMS Executive Director Elyse Gustafson for her quiet efficiency, willing support and generously provided advice when needed, and Sir David Cox for his ready encouragement as coordinating editor of the new IMS Monograph and Textbook series, in cooperation with Cambridge University Press.

We shall, of course, be most grateful for any brief alert (e.g. to mrl@email.unc.edu or pipiras@email.unc.edu) regarding remaining errors, blemishes or inelegance (which will exist a.s. in spite of years of revision!) as well as general reactions or comments a reader may be willing to share.