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THE THREE-DIMENSIONAL NAVIER–STOKES EQUATIONS

A rigorous but accessible introduction to the mathematical theory of the three-dimensional Navier–Stokes equations, this book provides self-contained proofs of some of the most significant results in the area, many of which can only be found in research papers. Highlights include the existence of global-in-time Leray–Hopf weak solutions and the local existence of strong solutions; the conditional local regularity results of Serrin and others; and the partial regularity results of Caffarelli, Kohn, and Nirenberg.

Appendices provide background material and proofs of some ‘standard results’ that are hard to find in the literature. A substantial number of exercises are included, with full solutions given at the end of the book. As the only introductory text on the topic to treat all of the mainstream results in detail, this book is an ideal text for a graduate course of one or two semesters. It is also a useful resource for anyone working in mathematical fluid dynamics.

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The Three-Dimensional Navier–Stokes Equations

Classical Theory

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Contents

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Preface

The purpose of this book is to provide a rigorous but accessible introduction to the mathematical theory of the three-dimensional Navier–Stokes equations, suitable for graduate students, by giving self-contained proofs of what we see as three of the most significant results in the area:

- the existence of global-in-time Leray–Hopf weak solutions, i.e. weak solutions that satisfy the strong energy inequality (Leray, 1934; Hopf, 1951); and the local-in-time existence of strong solutions;
- local regularity results due to Serrin (1962) and others: if

$$u \in L^r((a,b); L^s(U)), \qquad \frac{2}{r} + \frac{3}{s} \leq 1,$$

  then u is spatially smooth within U; and
- the partial regularity result of Caffarelli, Kohn, & Nirenberg (1982) that guarantees that the one-dimensional parabolic Hausdorff measure of the set of space–time singularities of any suitable weak solution is zero.

We end with a result that makes use of many of the properties of solutions that we prove throughout the book, the almost-everywhere uniqueness of the Lagrangian particle trajectories for suitable weak solutions.

We also treat the following topics that often fall into the category of ‘standard results’ but turn out to be hard to find in the required form:

- the Helmholtz–Weyl decomposition on $\mathbb{T}^3$;
- the L^p boundedness of the Leray projector on $\mathbb{T}^3$;
- estimates on the pressure in the periodic case;
- weak solutions as distributional solutions;
- smoothness and uniqueness for $u \in L^r(0,T; L^s)$, $\frac{2}{r} + \frac{3}{s} = 1$, $3 < s \leq \infty$;
- the local existence of solutions in $H^{1/2}$ and L^3 via energy estimates;

– local estimates for the velocity given the vorticity (via Biot–Savart);
– the validity of the local energy inequality; and
– local and maximal regularity results for solutions of the heat equation.

We assume knowledge of the basic language and results common in the rigorous study of PDEs such as provided by Evans (1998), Renardy & Rogers (2004), or Robinson (2001), among many.

Particularly in the earlier parts of the book, where much of the material is well known and has been presented many times, we have chosen not to clutter the exposition with frequent and exhaustive references. Instead historical discussion and suggestions for further reading are delayed until the Notes at the end of each chapter. After the Notes there is generally a selection of exercises, which either expand on material in the main text or contain steps that would interrupt the flow of an argument. Full solutions are given at the end of the book. This makes the first part of the book, which treats the classical existence, uniqueness, and regularity theory, particularly suitable for a first course on the Navier–Stokes equations.

There are many other books that cover some of the material here: our presentation has been particularly influenced by Chemin, Desjardins, Gallagher, & Grenier (2006); Constantin & Foias (1988); Doering & Gibbon (1995); Galdi (2000, 2011); and Temam (1977, 1983). We touch only briefly on the analysis of the equation in critical spaces, which has been the focus of much research in the last two decades; this topic is covered in detail in the books by Cannone (1995, see also his 2003 review article) and Lemarié-Rieusset (2002).

We would like to thank all our mentors, colleagues, and students, who over the years have fostered our interest in this subject. We would particularly like to acknowledge the academic support and friendship of John Gibbon, Charles Fefferman, and Grzegorz Łukaszewicz. We are very grateful to David McCormick, Wojciech Ożański, Benjamin Pooley, and Mikołaj Sierżęga, who read various drafts of the book and gave us many helpful comments.

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