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Introduction

This book focuses on direct and inverse problems for canonical differential systems of the form

$$\frac{d}{dx}u(x,\lambda) = i\lambda u(x,\lambda)H(x)J \quad \text{a.e. on } [0,d), \tag{1.1}$$

where $\lambda \in \mathbb{C}$, $J \in \mathbb{C}^{m\times m}$ is a **signature matrix**, i.e., $J^* = J$ and $J^*J = I_m$, and $H(x)$ is an $m \times m$ mvf (matrix-valued function) that is called the **Hamiltonian** of the system and is assumed to satisfy the conditions

$$H \in L_{1,\mathrm{loc}}^{m\times m}([0,d)) \quad \text{and} \quad H(x) \geq 0 \quad \text{a.e. on } [0,d). \tag{1.2}$$

The solution $u(x,\lambda)$ of (1.1) under condition (1.2) is a locally absolutely continuous $k \times m$ mvf that is uniquely defined by specifying an initial condition $u(0,\lambda)$; it is also the unique locally absolutely continuous solution of the integral equation

$$u(x,\lambda) = u(0,\lambda) + i\lambda \int_0^x u(s,\lambda)H(s)ds J, \quad 0 \leq x < d. \tag{1.3}$$

The signature matrix J that is usually considered in (1.1) and (1.3) is either $\pm I_m$ or $\pm j_{pq}$, $\pm J_p$ and $\pm \mathcal{J}_p$, where

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \; J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}, \; \mathcal{J}_p = \begin{bmatrix} 0 & -iI_p \\ iI_p & 0 \end{bmatrix} \tag{1.4}$$

and $p + q = m$ in the first displayed matrix and $2p = m$ in the other two. Every signature matrix $J \neq \pm I_m$ is unitarily similar to j_{pq}, where

$$p = \operatorname{rank}(I_m + J) \quad \text{and} \quad q = \operatorname{rank}(I_m - J) \quad \text{with } p + q = m. \tag{1.5}$$

Thus, $\pm J_p$ and $\pm \mathcal{J}_p$ are unitarily similar to $j_p = j_{pp}$:

$$J_p = \mathfrak{V}^* j_p \mathfrak{V}, \quad \text{where} \quad \mathfrak{V} = \frac{1}{\sqrt{2}}\begin{bmatrix} -I_p & I_p \\ I_p & I_p \end{bmatrix} \tag{1.6}$$

and

$$\mathcal{J}_p = \mathfrak{V}_1^* J_p \mathfrak{V}_1, \quad \text{where} \quad \mathfrak{V}_1 = \begin{bmatrix} -iI_p & 0 \\ 0 & I_p \end{bmatrix}. \tag{1.7}$$

Consequently a canonical system (1.1) with arbitrary signature matrix $J \neq \pm I_m$ may (and will) be reduced to a corresponding differential system with j_{pq} in place of J, or with J_p or $\mathcal{J}_p$ if $q = p$. The choice of J depends upon the problem under consideration. Thus, for example, $J = j_{pq}$ is appropriate for direct and inverse scattering problems, whereas the choice $J = J_p$ (or $J = \mathcal{J}_p$) is appropriate for direct and inverse spectral and input impedance (alias Weyl–Titchmarsh function) problems.

A number of different second-order differential equations and systems of differential equations of first-order may be reduced to the canonical differential system (1.1): the Feller–Krein string equation, the Dirac–Krein differential system, the Schrödinger equation with a matrix-valued potential of the form $q(x) = v'(x) \pm v(x)^2$ and Sturm–Liouville equations with appropriate restrictions on the coefficients.

The canonical system (1.1) arises (at least formally) by applying the Fourier transform

$$u(x, \lambda) = \widehat{y}(x, \lambda) = \int_0^\infty e^{i\lambda t} y(x, t)dt$$

to the solution $y(x, t)$ of the Cauchy problem

$$\begin{aligned} \frac{\partial y}{\partial x}(x, t) &= -\frac{\partial y}{\partial t}(x, t)H(x)J, \quad 0 \le x \le d, \ 0 \le t < \infty, \\ y(x, 0) &= 0. \end{aligned} \tag{1.8}$$

The $m \times m$ matrix-valued solution $U_x(\lambda)$ of the system (1.1) that satisfies the initial condition $U_0(\lambda) = I_m$ is called the **matrizant** of the system. It may be interpreted as the **transfer function** from the input data $y(0, t)$ to the output $y(x, t)$ on the interval $[0, x]$ of the system in which the evolution of the data is described by equation (1.8), since

$$\widehat{y}(x, \lambda) = \widehat{y}(0, \lambda)U_x(\lambda) \quad \text{for } 0 \le x < d. \tag{1.9}$$

1.1 The matrizant as a chain of entire J-inner mvf's

The matrizant $U_x(\lambda)$ is an entire mvf in the variable λ for each $x \in [0, d)$ such that

$$\frac{d}{dx}\left\{U_x(\lambda)JU_x(\omega)^*\right\} = i(\lambda - \overline{\omega})U_x(\lambda)H(x)U_x(\omega)^* \quad \text{a.e. on the interval } (0, d),$$

which in turn implies that

$$U_{x_2}(\lambda)JU_{x_2}(\omega)^* - U_{x_1}(\lambda)JU_{x_1}(\omega)^* = i(\lambda - \overline{\omega})\int_{x_1}^{x_2} U_x(\lambda)H(x)U_x(\omega)^* dx. \tag{1.10}$$

Thus, as $U_0(\lambda) = I_m$, the kernel

$$K_\omega^U(\lambda) = \begin{cases} \dfrac{J - U(\lambda)JU(\omega)^*}{-2\pi i(\lambda - \overline{\omega})} & if\ \lambda \neq \overline{\omega} \\ \dfrac{1}{2\pi i}\left(\dfrac{\partial U}{\partial \lambda}\right)(\overline{\omega})JU(\omega)^* & if\ \lambda = \overline{\omega} \end{cases} \tag{1.11}$$

with $U(\lambda) = U_x(\lambda)$ is positive on $\mathbb{C} \times \mathbb{C}$ in the sense that

$$\sum_{i,j=1}^n v_i^* K_{\omega_j}^U(\omega_i)v_j \geq 0 \tag{1.12}$$

for every choice of points $\omega_1, \ldots, \omega_n$ in $\mathbb{C}$ and vectors $v_1, \ldots, v_n$ in $\mathbb{C}^m$, since

$$\sum_{i,j=1}^n v_i^* K_{\omega_j}^U(\omega_i)v_j = \frac{1}{2\pi}\int_0^x \left(\sum_{i=1}^n v_i^* U_s(\omega_i)\right) H(s) \left(\sum_{j=1}^n v_j^* U_s(\omega_j)\right)^* ds,$$

by formula (1.10). Therefore, by the matrix version of a theorem of Aronszajn in [Arn50], there is an RKHS (**reproducing kernel Hilbert space**) $\mathcal{H}(U)$ with RK (**reproducing kernel**) $K_\omega^U(\lambda)$ defined by formula (1.11); see Section 4.6.

Moreover, $U_x(\lambda)$ belongs to the class $\mathcal{E} \cap \mathcal{U}(J)$ of entire mvfs that are J-inner with respect to the open upper half plane $\mathbb{C}_+$:

$$U_x(\lambda)^* JU_x(\lambda) \leq J \quad \text{for } \lambda \in \mathbb{C}_+ \tag{1.13}$$

and

$$U_x(\lambda)^* JU_x(\lambda) = J \quad \text{for } \lambda \in \mathbb{R} \tag{1.14}$$

for every $x \in [0, d)$, and, as follows easily from (1.10) with $x_2 = x$ and $x_1 = 0$ (since $U_0(\lambda) \equiv I_m$),

$$U_x^\#(\lambda)JU_x(\lambda) = J \quad \text{for every point } \lambda \in \mathbb{C}, \tag{1.15}$$

in which

$$f^\#(\lambda) = f(\overline{\lambda})^*$$

for any mvf f that is defined at $\overline{\lambda}$.

The matrizant $U_x(\lambda)$, $0 \leq x < d$, is **nondecreasing** with respect to x in the sense that

$$U_{x_1}^{-1}U_{x_2} \in \mathcal{E} \cap \mathcal{U}(J) \quad \text{when } 0 \leq x_1 \leq x_2 < d. \tag{1.16}$$

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It is also locally absolutely continuous in $[0, d)$ with respect to x and normalized by the conditions

$$U_x(0) = I_m \quad \text{for} \quad 0 \leq x < d \quad \text{and} \quad U_0(\lambda) = I_m.$$

The symbol $\mathcal{U}(J)$ will denote the class of **J-inner** mvf's with respect to $\mathbb{C}_+$. This is the class of $m \times m$ mvf's that are: (1) meromorphic in $\mathbb{C}_+$; (2) meet the constraint (1.13) at points of holomorphy in $\mathbb{C}_+$; (3) meet the constraint (1.14) a.e. on $\mathbb{R}$ for the nontangential limits; and (4) are extended into the open lower half plane $\mathbb{C}_-$ by (1.15) with U in place of U_x; this class will be considered in more detail in Section 3.8.

Let

$\mathfrak{h}_f$ denote the set of points in $\mathbb{C}$ at which the mvf f is holomorphic

and let

$$\mathcal{U}^\circ(J) = \{U \in \mathcal{U}(J) : 0 \in \mathfrak{h}_U \quad \text{and} \quad U(0) = I_m\}. \tag{1.17}$$

Thus, the matrizant $U_x \in \mathcal{E} \cap \mathcal{U}^\circ(J)$ for every $x \in [0, d)$.

1.2 Monodromy matrices of regular systems

A system (1.1) with Hamiltonian H subject to (1.2) is said to be a **regular canonical differential system** if

$$d < \infty \quad \text{and} \quad H \in L_1^{m\times m}([0, d]). \tag{1.18}$$

In this case the solutions $u(x, \lambda)$ of the system (1.1) are considered on the closed interval $[0, d]$ and $u(x, \lambda)$ is absolutely continuous with respect to x on $[0, d]$. Thus, the value $U_d(\lambda)$ of the matrizant $U_x(\lambda)$ at the right-hand end point of the interval is well defined; it is called the **monodromy matrix** of the canonical differential system (1.1).

The monodromy matrix $U_d(\lambda)$ belongs to the class $\mathcal{E} \cap \mathcal{U}^\circ(J)$. It is a significant frequency characteristic of a regular system in which the evolution of the data $y(x, t)$ on the interval $0 \leq x \leq d$ is described by (1.8). The property (1.14) means that the system is **lossless**. In particular, equation (1.8) with $J = J_p$ describes the evolution $y(x, t) = [v(x, t),\ i(x, t)]$ of voltages $v(x, t) = [v_1(x, t), \ldots, v_p(x, t)]$ and currents $i(x, t) = [i_1(x, t), \ldots, i_p(x, t)]$ of a lossless ideal linear circuit that is realized by p-lines with distributed parameters

$$H(x) = \begin{bmatrix} C(x) & G(x) \\ G(x)^* & L(x) \end{bmatrix} \quad \text{for } 0 \leq x \leq d,$$

where $C(x)$, $G(x)$ and $L(x)$ are the distributed capacitance, gyratance and inductance, respectively. Thus, the **inverse monodromy problem**, which is to recover the Hamiltonian $H(x)$, given a monodromy matrix $U \in \mathcal{E} \cap \mathcal{U}^\circ(J)$, is a significant inverse problem for lossless p-line circuits with distributed parameters (for $J = J_p$).

A fundamental theorem of V.P. Potapov [Po60] guarantees that the inverse monodromy problem has at least one solution $H(x)$ that satisfies the normalization conditions

$$H \in L_1^{m\times m}([0,d]), \quad H(x) \geq 0 \quad \text{and} \quad \operatorname{trace} H(x) = 1 \text{ a.e. on } [0,d]. \tag{1.19}$$

A Hamiltonian $H(x)$ on an interval $[0, d]$ is said to be a **normalized Hamiltonian** if it meets the three conditions in (1.19). However, there may be more than one such solution $H(x)$.

If $J = I_m$, then $\mathcal{U}(J)$ coincides with

$$\mathcal{S}_{\text{in}}^{m\times m} = \{\text{the class of } m \times m \text{ inner functions with respect to } \mathbb{C}_+\}$$

and the inverse monodromy problem for a given monodromy matrix $U \in \mathcal{E} \cap \mathcal{S}_{\text{in}}^{m\times m}$ with $U(0) = I_m$ has a unique solution $H(x)$ subject to the conditions in (1.19) if and only if $U(\lambda)$ and $\det U(\lambda)$ have the same exponential type. This criterion, which is due to Brodskii and Kisilevskii (see, e.g., [Bro72]), will be discussed in Chapter 8.

There are also uniqueness theorems for the inverse monodromy problem when $J \neq \pm I_m$, under extra conditions on the given monodromy matrix and the unknown Hamiltonian $H(x)$: If $J = \mathcal{J}_p$ and the monodromy matrix is **symplectic** (i.e., $U(\lambda)^\tau \mathcal{J}_p U(\lambda) = \mathcal{J}_p$, where $U(\lambda)^\tau$ denotes the transpose of $U(\lambda)$) and $p = 1$, then, by a fundamental theorem of L. de Branges [Br68a], there exists exactly one real solution $H(x) = \overline{H(x)} \geq 0$ a.e. on $[0, d]$ of the inverse monodromy problem that meets the normalization condition (1.19). This theorem and a number of its implications (including a uniqueness theorem for the inverse monodromy problem for the Feller–Krein string equation) will be discussed in Chapter 2 and again, in more detail, in Chapter 8.

1.3 Canonical integral systems

Every canonical differential system (1.1) with Hamiltonian $H(x)$ that satisfies (1.2) is equivalent to the **canonical integral system**

$$u(x,\lambda) = u(x,0) + i\lambda \int_0^x u(s,\lambda)dM(s)J, \quad 0 \leq x < d, \tag{1.20}$$

with

$$M(x) = \int_0^x H(s)ds.$$

We shall consider the system (1.20) under less restrictive conditions on $M(x)$, i.e., assuming only that

$$M(x) \quad \text{is a continuous nondecreasing} \\ m \times m \text{ mvf on } [0, d) \text{ with } M(0) = 0. \tag{1.21}$$

The continuous solution $U_x(\lambda) = U(x, \lambda)$ of the system (1.20) with $U_0(\lambda) = I_m$ is called the **matrizant** of the system. In view of (1.21), the matrizant satisfies an analog of (1.10) with $dM(x)$ in place of $H(x)dx$. Thus, the matrizant is a family of entire J-inner mvf's with respect to λ that is continuous and nondecreasing with respect to x on the interval $[0, d)$. Moreover, U_x, $0 \le x < d$, is a normalized family, since $U_x \in \mathcal{U}^\circ(J)$ for $0 \le x < d$ and $U_0(\lambda) = I_m$.

If $d < \infty$ and $M(x)$ is bounded on $[0, d)$, then $M(x)$ extends continuously to $[0, d]$ and the continuous solution $U_x(\lambda)$ of the system (1.20) is considered on $[0, d]$. The system (1.20) is a said to be a **regular canonical integral system** if

$$d < \infty \quad \text{and} \quad M(x) \text{ is a continuous nondecreasing} \\ m \times m \text{ mvf on } [0, d] \text{ with } M(0) = 0. \tag{1.22}$$

The matrizant $U_x(\lambda)$ of a regular integral system is a normalized nondecreasing continuous family of entire J-inner mvf's on the closed interval $[0, d]$; the mvf $U_d(\lambda)$ is called the **monodromy matrix** of the system.

1.4 Singular, right regular and right strongly regular matrizants

The main results on direct and inverse problems for canonical integral systems that are presented in this monograph are obtained for the class of systems with matrizants that satisfy the extra condition

$$U_x \in \mathcal{U}_{rR}(J) \quad \text{for } 0 \le x < d, \tag{1.23}$$

in which $\mathcal{U}_{rR}(J)$ denotes the class of **right regular** J-inner mvf's $U(\lambda)$ that may be defined in terms of the RKHS $\mathcal{H}(U)$ as

$$\mathcal{U}_{rR}(J) = \{U \in \mathcal{U}(J) : \mathcal{H}(U) \cap L_2^m(\mathbb{R}) \quad \text{is dense in } \mathcal{H}(U)\}. \tag{1.24}$$

Existence and uniqueness will be obtained for the solutions of a number of inverse problems in the class of systems with matrizants that meet the condition (1.23).

Formulas for U_x and $M(x)$ will be obtained under the more restrictive condition

$$U_x \in \mathcal{U}_{rsR}(J) \quad \text{for } 0 \le x < d, \tag{1.25}$$

where $\mathcal{U}_{rsR}(J)$ denotes the class of **right strongly regular** J-inner mvf's that may be defined in terms of the RKHS $\mathcal{H}(U)$ as

$$\mathcal{U}_{rsR}(J) = \{U \in \mathcal{U}(J) : \mathcal{H}(U) \subset L_2^m(\mathbb{R})\}. \tag{1.26}$$

If $U \in \mathcal{E} \cap \mathcal{U}(J)$, then $\mathcal{H}(U)$ is a Hilbert space of entire $m \times 1$ vvf's (vector-valued functions) f. The condition in (1.26) means that the restriction of $f \in \mathcal{H}(U)$ to $\mathbb{R}$ belongs to $L_2^m(\mathbb{R})$.

The class

$$\mathcal{U}_{AR}(J) = \{U \in \mathcal{U}(J) : \text{ every left divisor of } U \text{ belongs to the class } \mathcal{U}_{rR}(J)\}$$

that sits between $\mathcal{U}_{rR}(J)$ and $\mathcal{U}_{rsR}(J)$:

$$\mathcal{U}_{rsR}(J) \subset \mathcal{U}_{AR}(J) \subset U_{rR}(J),$$

will also play a significant role; it is discussed in detail in Section 4.9.

In Chapter 12 it will be shown that the matrizants of Dirac–Krein systems with locally summable potentials belong to the class $\mathcal{U}_{rsR}(J)$. On the other hand, the matrizant $Y_x(\lambda)$, $0 \le x < d$, of a Schrödinger equation

$$-y''(x,\lambda) + y(x,\lambda)q(x) = \lambda y(x,\lambda), \quad 0 \le x < d,$$

with potential

$$q \in L_{1,\mathrm{loc}}^{p\times p}([0,d)) \quad \text{and} \quad q(x) = q(x)^* \text{ a.e. on } [0,d)$$

belongs to the class

$$\mathcal{U}_S(J) = \{U \in \mathcal{U}(J) : \mathcal{H}(U) \cap L_2^m(\mathbb{R}) = \{0\}\} \tag{1.27}$$

of **singular** J-inner mvf's. Nevertheless, if the Riccati equation

$$q(x) = v'(x) + v(x)^2 \quad \text{or} \quad q(x) = v'(x) - v(x)^2$$

admits a locally summable solution $v(x)$ on $[0,d)$, then the Schrödinger equation may be reduced to the Dirac system

$$\frac{d}{dx}u(x,\lambda) = i\lambda u(x,\lambda)j_p + u(x,\lambda)\begin{bmatrix} 0 & v(x) \\ v(x)^* & 0 \end{bmatrix}, \quad 0 \le x < d, \tag{1.28}$$

and the matrizant of this system satisfies the condition (1.25); see Chapter 12 for the details.

In Chapter 5 it will be shown that any continuous normalized nondecreasing family of entire J-inner mvf's $U_x(\lambda)$, $0 \le x < d$, that satisfies the condition (1.23) is the matrizant of exactly one canonical integral system (1.20) with $M(x)$ satisfying (1.21). Moreover,

$$M(x) = 2\pi K_0^{U_x}(0) = \lim_{\lambda \to 0} \frac{U_x(\lambda) - I_m}{i\lambda} J = -i\frac{\partial U_x}{\partial \lambda}(0)J \tag{1.29}$$

for every system (1.20) with a mass function $M(x)$ that satisfies (1.21); see Theorem 5.8.

In view of the characterization (1.26) and the fact that

$$\mathcal{H}(U_x) \subseteq \mathcal{H}(U_d) \quad \text{for every } x \in [0,d],$$

it follows that

$$U_d \in \mathcal{U}_{rsR}(J) \Longrightarrow U_x \in \mathcal{U}_{rsR}(J) \quad \text{for every } x \in [0, d].$$

Moreover, if $U_x(\lambda)$, $0 \le x \le d$, is a nondecreasing continuous normalized family of entire J inner mvf's and $U_d \in \mathcal{U}_{rsR}(J)$, then $U_x(\lambda)$, $0 \le x \le d$, is the matrizant of exactly one canonical integral system on $[0, d]$. This system is regular; U_d is its monodromy matrix and formula (1.29) holds on the closed interval $[0, d]$.

Since

$$L_\infty^{m\times m}(\mathbb{R}) \cap \mathcal{U}(J) \subset \mathcal{U}_{rsR}(J),$$

a monodromy matrix U belongs to the class

$$U \in \mathcal{E} \cap \mathcal{U}_{rsR}^\circ(J) \tag{1.30}$$

if it is bounded on $\mathbb{R}$. Thus,

$$\mathcal{S}_{\text{in}}^{m\times m} = \mathcal{U}_{rsR}(I_m).$$

The condition (1.30) on a monodromy matrix does not guarantee the uniqueness of a normalized solution $H(x)$ for the inverse monodromy problem even when $J = I_m$ (in view of the Brodskii–Kisilevskii criterion).

In this book, we shall focus attention on direct and inverse problems for canonical integral and differential systems with $J \neq \pm I_m$ with matrizants that meet the condition (1.23). In this case it may be assumed that $J = j_{pq}$ or $J = J_p$, if $q = p$.

The matrizant of a canonical integral system (1.20) will be denoted by $W_x(\lambda)$ if $J = j_{pq}$ and by $A_x(\lambda)$ if $J = J_p$. The matrizant $W_x(\lambda)$ (resp., $A_x(\lambda)$) will play an important role in the study of direct and inverse input scattering (resp., input impedance and spectral) problems.

1.5 Input scattering matrices

If $W \in \mathcal{U}(j_{pq})$, then the **linear fractional transformation**

$$T_W[\varepsilon] = (w_{11}\varepsilon + w_{12})(w_{21}\varepsilon + w_{22})^{-1} \tag{1.31}$$

based on the four-block decomposition

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \tag{1.32}$$

with blocks w_{11} of size $p \times p$ and w_{22} of size $q \times q$ maps the **Schur class**

$$\mathcal{S}^{p\times q} = \{\text{holomorphic contractive } p \times q \text{ mvf's } \varepsilon \text{ in } \mathbb{C}_+\} \tag{1.33}$$

into itself, i.e.,

$$T_W[\mathcal{S}^{p\times q}] \stackrel{\text{def}}{=} \{T_W[\varepsilon] : \varepsilon \in \mathcal{S}^{p\times q}\} \subseteq \mathcal{S}^{p\times q}. \tag{1.34}$$

Since the matrizant W_x, $0 \le x < d$, is a nondecreasing family of j_{pq}-inner mvf's, this inclusion implies that $T_{W_{x_2}}[\mathcal{S}^{p\times q}] \subseteq T_{W_{x_1}}[\mathcal{S}^{p\times q}]$ if $0 \le x_1 \le x_2 < d$ and that the set

$$\mathcal{S}^d_{\rm scat}(dM) \stackrel{\rm def}{=} \bigcap_{0\le x<d} T_{W_x}[\mathcal{S}^{p\times q}] \tag{1.35}$$

is a nonempty subset of $\mathcal{S}^{p\times q}$. The mvf's s that belong to $\mathcal{S}^d_{\rm scat}(dM)$ will be called **input scattering matrices** of the system (1.20) for $J = j_{pq}$. A physical interpretation of this notion is presented in Section 6.1 for regular systems, in which case

$$\mathcal{S}^d_{\rm scat}(dM) = T_{W_d}[\mathcal{S}^{p\times q}].$$

There is another characterization of the class $\mathcal{U}_{rsR}(J)$ in addition to (1.26) when $J = j_{pq}$:

$$W \in \mathcal{U}_{rsR}(j_{pq}) \Longleftrightarrow T_W[\mathcal{S}^{p\times q}] \cap \mathring{\mathcal{S}}^{p\times q} \ne \emptyset, \tag{1.36}$$

where

$$\mathring{\mathcal{S}}^{p\times q} = \{s \in \mathcal{S}^{p\times q} : \ \|s\|_\infty < 1\}. \tag{1.37}$$

Thus, the matrizant W_x, $0 \le x < d$, of every solution of the inverse input scattering problem with $s \in \mathring{\mathcal{S}}^{p\times q}$ will be a family of strongly right regular j_{pq}-inner mvf's.

1.6 Chains of associated pairs of the first kind

The set of canonical integral systems (1.20) with $J = j_{pq}$ and a given input scattering matrix $s \in \mathcal{S}^{p\times q}_{sz}$ that is defined in (1.56) (and the set of regular canonical systems with a given monodromy matrix $W \in \mathcal{E} \cap \mathcal{U}^\circ_{rR}(j_{pq})$) with matrizant

$$W_x \in \mathcal{U}^\circ_{rR}(j_{pq}) \quad \text{for } 0 \le x < d \tag{1.38}$$

will be parametrized in terms of a **chain** $\{b_1^x, b_2^x\}$, $0 \le x < d$, **of pairs** of entire inner mvf's $b_1^x \in \mathcal{S}^{p\times p}_{\rm in}$, $b_2^x \in \mathcal{S}^{q\times q}_{\rm in}$ that is

nondecreasing in the sense that

$$(b_1^{x_1})^{-1} b_1^{x_2} \in \mathcal{E} \cap \mathcal{S}^{p\times p}_{\rm in} \quad \text{and} \quad b_2^{x_2}(b_2^{x_1})^{-1} \in \mathcal{E} \cap \mathcal{S}^{q\times q}_{\rm in}, \quad 0 \le x_1 \le x_2 < d;$$

continuous in the sense that $b_1^x(\lambda)$ and $b_2^x(\lambda)$ are continuous mvf's of x on $[0, d)$ for each fixed $\lambda \in \mathbb{C}$; and

normalized by the condition

$$b_1^x(0) = I_p, \quad b_2^x(0) = I_q \quad \text{for } 0 \le x < d, \quad b_1^0(\lambda) \equiv I_p \quad \text{and} \quad b_2^0(\lambda) \equiv I_q.$$

This parametrization rests on the association of a pair of inner mvf's $\{b_1, b_2\}$ to each $W \in \mathcal{U}(j_{pq})$ that is obtained from the inner–outer factorization and outer–inner factorizations of the mvf's $(w_{11}^{\#})^{-1}$ and w_{22}^{-1}, which belong to $\mathcal{S}^{p\times p}$ and $\mathcal{S}^{q\times q}$, respectively:

$$(w_{11}^{\#})^{-1} = b_1\varphi_1 \quad \text{and} \quad w_{22}^{-1} = \varphi_2 b_2, \tag{1.39}$$

where

$$b_1 \in \mathcal{S}_{\text{in}}^{p\times p},\ b_2 \in \mathcal{S}_{\text{in}}^{q\times q},\ \varphi_1 \in \mathcal{S}_{\text{out}}^{p\times p},\ \varphi_2 \in \mathcal{S}_{\text{out}}^{q\times q}$$

and

$$\mathcal{S}_{\text{out}}^{p\times p} = \{\text{the set of outer mvf's in } \mathcal{S}^{p\times p}\}.$$

Such a pair $\{b_1, b_2\}$ will be called an **associated pair** of W and the set of such pairs will be denoted $ap(W)$. In view of the implication

$$W \in \mathcal{E}\cap\mathcal{U}(j_{pq}) \quad \text{and} \quad \{b_1, b_2\} \in ap(W) \Longrightarrow \\ b_1 \in \mathcal{E}\cap\mathcal{S}_{\text{in}}^{p\times p} \text{ and } b_2 \in \mathcal{E}\cap\mathcal{S}_{\text{in}}^{q\times q}, \tag{1.40}$$

the pair $\{b_1, b_2\}$ may be uniquely specified by the normalization conditions $b_1(0) = I_p$ and $b_2(0) = I_q$ when $W \in \mathcal{E}\cap\mathcal{U}(j_{pq})$.

The converse of the implication (1.40) is valid when W belongs to the class $\mathcal{U}_{rR}(j_{pq})$ of right regular j_{pq}-inner mvf's, but not in general.

Since $W_x(\lambda)$, $0 \le x < d$, is a nondecreasing family, there is a unique normalized nondecreasing chain of pairs of entire inner mvf's $\{b_1^x, b_2^x\}$, $0 \le x < d$, corresponding to the matrizant $W_x(\lambda)$, $0 \le x < d$, of the system (1.20) with $J = j_{pq}$ and $M(x)$ subject to the restrictions in (1.21) such that

$$\{b_1^x, b_2^x\} \in ap(W_x) \quad \text{for } 0 \le x < d. \tag{1.41}$$

If also

$$W_x \in \mathcal{U}_{rR}(j_{pq}) \quad \text{for } 0 \le x < d, \tag{1.42}$$

then, as will be shown in Theorem 5.13, the chain of pairs is continuous too. Consequently, if (1.42) is in force, then there exists exactly one continuous normalized nondecreasing chain of pairs $\{b_1^x, b_2^x\}$ of entire inner mvf's such that (1.41) holds.

Associated pairs are also defined for J-inner mvf's U with

$$J = V_1^* j_{pq} V_1 \quad \text{for some } m\times m \text{ unitary matrix } V_1 \tag{1.43}$$

upon noting that if

$$W(\lambda) = V_1 U(\lambda) V_1^*, \tag{1.44}$$